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**On Some Analogues of Lack of Memory Properties for the Gompertz Distribution**

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**Abstract:** We discuss some analogues of the lack of memory property, the strong lack of memory property and the almost lack of memory properties for the Gompertz distribution as applications of the integrated Cauchy functional equation.

**Mathematics Subject Classification:** 60E05, 62E10.

**Keywords**: Characterization; Exponential distribution; Gompertz distribution; Integrated Cauchy functional equation; Lack of memory property; Strong lack of memory property; Almost lack of memory property.

# 1 Introduction

Gompertz distribution is a continuous prbability distribution named after Benjamin Gompertz. It is often used to model the distribution of adult lifespans (cf. Vaupel (1986); Preston et al. (2001); Benjamin et al. (1980) and Willemse and Koppelar (2000)). It is also used as a mathematical model of aging processes by Brown and Forbes (1974) and for studying the rate of aging in Economos (1982). More recently, computer scientists have used it to model the failure rate of software codes (Ohishi et al.(2009)). In management science, Gompertz distribution has been used for modeling purchase behavior with sudden "death" as a flexible customer lifetime model. Pollard and Valkovics (1992) discuss some properties and applications of the Gompertz distribution.

Suppose $X$ is a non-negative continuous random variable with Gompertz distribution having the survival function

$$S(x) = \exp[-a(e^{bx} - 1)], x \geq 0.$$

It is easy to check that the function $S(x)$ satisfies the functional equation

$$\frac{S(x+t)}{S(t)} = [S(x)]^{\xi(t)}, x \geq 0, t \geq 0$$

with $\xi(t) = e^{bt}$. Conversely suppose $X$ is any non-negative continuous random variable with survival function $S(.) = P(X > x)$ satisfying the functional equation

$$\frac{S(x+t)}{S(t)} = [S(x)]^{\xi(t)}, x \geq 0, t \geq 0 \tag{1. 1}$$

where $\xi : [0, \infty) \to [0, \infty)$. Kaminsky (1983) proved that the random variable $X$ must then have the Gompertz distribution with the survival function

$$S(x) = \exp[-a(e^{bx} - 1)], x \geq 0 \tag{1. 2}$$

for some constants $a > 0$ and $b > 0$ and the function $\xi(t) = e^{bt}, t \geq 0$. Marshall and Olkin (2007) obtained characterizations of distributions through coincidences of semiparametric families and these characterization results include a characterization of the Gompertz distribution. Marshall and Olkin (2015) derived a bivariate Gompertz-Makeham distribution based on a bivariate version of the functional equation (1.1). Kolev (2016) obtained characterizations of the class of bivariate Gompertz distributions introduced by Marshall and Olkin (2015). The following theorem, as given by Kolev (2016), is due to Kaminsky (1983)(cf. Marshall and Olkin (2007)).

**Theorem 1.1:** Suppose that $X$ is an absolutely continuous non-negative random variable such that its survival function $S(x) > 0$ for all $x > 0$. Then the function $S(.)$ satisfies the equation (1.1) for some $b > 0$ and for some function $\xi(.)$ not depending on $x$ if and only if either

(i) the function $S(x) = e^{-bx}$ and $\xi(t) \equiv 1$; or
(ii) the function $S(x)$ is the survival function of the Gompertz distribution given by the equation (1.2) and $\xi(t) = e^{bt}$; or
(iii) the function $S(x) = \exp[a(e^{-bx} - 1)]$ and $\xi(t) = e^{-bt}$.

The exponential and the Gompertz survival functions are solutions of the Kaminsky functional equation given by (1.1). The negative Gompertz distribution, given by part (iii) in Theorem 1.1, is also a solution of the functional equation (1.1) but the corresponding function $S(x)$ does not give a proper probability distribution function as pointed out by Kolev (2016).

## 2 Preliminaries

Let $X$ be a non-negative absolutely continuous random variable with the survival function $S(.)$. The equation (1.1) can be written in an alternative form

$$P(X > x + t | X > t) = [P(X > x)]^{\xi(t)}, x \geq 0, t \geq 0. \tag{2. 1}$$

If the function $\xi(t) \equiv 1$, then the equation (2.1) characterizes the exponential distribution as is well known and pointed out in Theorem 1.1. This is also known as the "lack of memory" property of the exponential distribution. It is known the exponential distribution has the "strong" lack of memory property in the following sense: Suppose a random variable $X$ has the exponential distribution. Then, for any non-negative random variable $Y$ indepenedent of $X$,

$$P(X > Y + x | X > Y) = P(X > x), x \geq 0. \tag{2. 2}$$

It is known that, if the equation (2.2) holds for any two independent non-negative random variables $X$ and $Y$, then the random variable $X$ will have an expoential distribution under some conditions (cf. Theorem 2.5.1, Ramachandran and Lau(1991), p.40). The equation (2.2) leads to what is known as "Integrated Cauchy Functional Equation". For an extensive discussion on such equations and their applications to characterizations of probability distributions, see Ramachandran and Lau (1991) and Rao and Shanbhag (1994). Let $G(.)$ be the distribution function of the random variable $Y$ in the equation (2.2). Since the random variables $X$ and $Y$ are independent, the equation (2.2) reduces to

$$\int_0^\infty S(x+y)G(dy) = c\, S(x), x \geq 0 \tag{2. 3}$$

where $c = P(X > Y)$ or equivalently

$$\int_0^\infty [S(x+y) - S(x)S(y)]G(dy) = 0, x \geq 0. \tag{2. 4}$$

The following result holds (cf. Theorem 2.5.1, Lau and Ramachandran (1991)).

**Theorem 2.1:** Suppose that $G(0) < c < 1$. Let $\lambda > 0$ be defined by the equation

$$\int_0^\infty e^{-\lambda y} G(dy) = c \tag{2. 5}$$

and let, for $\rho > 0, A(\rho) = \{n\rho, n \geq 1\}$. Then
(i) $S(x) = e^{-\lambda x}, x \geq 0$ if the support of the distribution function $G(.)$ is not contained in the set $A(\rho)$ for any $\rho > 0$; and

(ii) $S(x) = p(x)e^{-\lambda x}, x \geq 0$ where the function $p(.)$ is right continuous and has period $\rho$ if the support of the distribution function $G(.)$ is contained in the set $A(\rho)$ for some $\rho > o$ which is taken to be the largest such value.

## 3 Main Results

In analogy with the equation which follows from the "strong" lack of memory property of the exponential distribution, we now consider the following functional equation

$$\frac{S(x+Y)}{S(Y)} = [S(x)]^{\xi(Y)}, x \geq 0 \tag{3. 1}$$

where $Y$ is a non-negative random variable independent of the non-negative random variable $X$ with the survival function $S(.)$ Our aim is to characterize the family of survival functions $S(.)$ and the functions $\xi(.)$ satisfying the equation (3.1). Let $R(x) = -\log S(x)$. From the independence of the random Variables $X$ and $Y$, the equation (3.1) can be written in the form

$$\int_0^\infty [R(x+y) - R(y) - \xi(y)R(x)]G(dy) = 0, x \geq 0 \tag{3. 2}$$

Suppose that the function $R(.)$ is differentiable with derivative $R'(.)$. Further suppose that differentiation under integral sign with respect to $x$ is permitted in the equation (3.2). Then, differentiating under the integral sign with respect to $x$, the equation (3.2) leads to the equation

$$\int_0^\infty [R'(x+y) - \xi(y)R'(x)]G(dy) = 0, x \geq 0. \tag{3. 3}$$

Let $x = 0$ in the equation (3.3). It follows that

$$\int_0^\infty [R'(y) - \xi(y)R'(0)]G(dy) = 0 \tag{3. 4}$$

which implies that

$$\int_0^\infty R'(y)G(dy) = R'(0)\int_0^\infty \xi(y)G(dy). \tag{3. 5}$$

Applying this relation in the equation (3.3), we get that

$$\int_0^\infty [R'(x+y) - R'(0)\ R'(x)R'(y)]G(dy) = 0, x \geq 0 \tag{3. 6}$$

or equivalently

$$\int_0^\infty R'(x+y)G(dy) = c\ R'(x), x \geq 0 \tag{3. 7}$$

for some constant $c$. This is an integrated Cauchy functional equation. Suppose $G(0) < 1$. Applying Theorem 2.2.4 in Ramachandran and Lau(1991), it follows that

$$R'(x) = p(x)e^{\lambda x} \text{ a.e.} \tag{3. 8}$$

where $\lambda \in R$ and is uniquely determined by the equation

$$\int_0^\infty e^{\lambda y} G(dy) = 1 \tag{3. 9}$$

and the function $p(.)$ satisfies $p(x+y) = p(x)$ for all $y$ in the support of the distribution function $G(.)$.

We have the following result.

**Theorem 3.1:** Suppose $X$ and $Y$ are independent non-negative random variables satisfying the functional equation (3.1) based on a function $\xi(.)$. Further suppose that the function the survival function $S(.)$ is differentiable and differentiation under integral sign is allowed in the integral equation corresponding to the equation (1.1). Then

$$R'(x) = -\log S(x) = p(x)e^{-\lambda x}, x \geq 0 \tag{3. 10}$$

where $\lambda \in R$ and is uniquely determined by the equation

$$\int_0^\infty e^{\lambda y} G(dy) = 1 \tag{3. 11}$$

and the function $p(.)$ satisfies $p(x+y) = p(x)$ for all $y$ in the support of the distribution function $G(.)$.

**Special cases:** (i)Suppose the support of the distribution function $G$ is $R_+$. then the function $p(.)$ is a constant, say, $c$ and

$$R'(x) = ce^{\lambda x}, x \geq 0 \tag{3. 12}$$

which implies that

$$R(x) = c(e^{\lambda x} - 1), x \geq 0 \tag{3. 13}$$

where $\lambda$ is given by the equation

$$\int_0^\infty e^{\lambda y} G(dy) = 1 \tag{3. 14}$$

Since the function $R(x) = -\log S(x)$ where $S(.)$ is the survival function of a random variable $X$, it is easy to see that the parameters $c$ and $\lambda$ are positive and the the distribution of the random variable $X$ is the Gompertz distribution with

$$S(x) = \exp[-a(e^{\lambda x} - 1))], x \geq 0 \tag{3. 15}$$

where $a$ and $\lambda$ are positive constants.

(ii) Suppose the functional equation (3.1) holds for a non-negative random variable $Y$ with distribution function $G$ with support not equal to R. It follows, from the general results on Cauchy functional equation that the support has to be a set which is a subset of $A_d = \{nd : n \geq 0\}$ where $d$ is the period of the function $p(x)$ and

$$R'(x) = p(x)e^{\lambda x}, x \geq 0. \tag{3. 16}$$

(cf. Marsagalia and Tubilla (1975)). Note that the functional equation (3.1) reduces to

$$\frac{P(X > x + nd)}{P(X > nd)} = [P(X > x)]^{\xi(nd)}, n \geq 0 \tag{3. 17}$$

Suppose the function $\xi(t) \equiv 1$. Then the above equation reduces to

$$\frac{P(X > x + nd)}{P(X > nd)} = P(X > x), n \geq 0. \tag{3. 18}$$

Distributions satisfying this functional equation are called distributions with periodic failure rate and have been studied in Prakasa Rao (1997). The same class under the terminology of "Almost lack of memory" property have been discussed in Chukova and Dimitrov (1992). We will now investigate similar properties of the functional equation (3.16).

Suppose that $X$ is a non-negative random variable satisfying the functional equation (3.17). Then

$$R'(x) = p(x)e^{\lambda x}, x \geq 0 \tag{3. 19}$$

where $p(.)$ is a non-negative periodic function with period $d$. Note that the function $R'(.)$ is the hazard rate or the failure rate of the random variable $X$. In particular,

$$R(x) = -\log S(x) = \int_0^x p(y)e^{\lambda y} dy, x \geq 0 \tag{3. 20}$$

where $p(.)$ is a periodic function with some period $d > 0$. Hence, for any $0 \leq x \leq d$,

$$\int_{nd}^{x+nd} p(y)e^{\lambda y} dy = \int_0^x p(z + nd)e^{\lambda(z+nd)} dz = e^{\lambda nd} \int_0^x p(z)e^{\lambda z} dz$$

which implies that

$$R(x+nd) - R(nd) = e^{\lambda nd} R(x), 0 \leq x \leq d, n \geq 0$$

or equivalently

$$-\log S(x+nd) + \log S(nd) = -e^{\lambda nd} \log S(x), 0 \leq x \leq d, n \geq 0$$

Hence

$$S(x+nd) = S(nd)[S(x)]^{e^{\lambda nd}}, 0 \leq x \leq d, n \geq 1. \quad (3.\ 21)$$

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