CRRAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)

Research Report

Author (s): S. B. Rao, U. K. Sahoo and V. Parameswaran

Title of the Report: **Cycle Stochastic Graphs: A Structural Characterization**

Research Report No.: **RR2015-09**

Date: **June 30, 2015**

Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org

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Cycle Stochastic Graphs: A Structural Characterization

S. B. Rao ·
U. K. Sahoo* ·
V. Parameswaran

Received: date / Accepted: date June 29, 2015

Abstract A vertex (edge) [vertex edge] cycle stochastic function of a graph G is a labeling of vertices (edges) [vertices and edges] by non-negative real valued function $f_V : V(G) \to \mathbb{R}^+$ ($f_E : E(G) \to \mathbb{R}^+$) [$f_{VE} : V(G) \cup E(G) \to \mathbb{R}^+$] such that for every cycle of G, the sum of labels of its vertices (edges) [vertices and edges] is 1. We begin by proving vertex edge cycle stochastic graphs are same as edge cycle stochastic graphs. We then find the following: (1) structure theorem for biconnected vertex cycle stochastic graphs and edge cycle stochastic graphs, (2) a minimal forbidden graph characterization for biconnected vertex cycle stochastic graphs, (3) description of a minimal forbidden graph characterization for vertex cycle stochastic graphs, and (4) some graph characteristics and algorithms to find them when restricted to these classes of graphs.

Keywords cycle stochastic graphs · edge cycle stochastic · vertex cycle stochastic · structural characterization · forbidden induced subgraphs · computational complexity

1 Introduction

Motivation: Berge introduced the notion of stochastic graphs in [2] as a generalization of strongly perfect graphs i.e. graphs whose every induced subgraph contains an independent set of vertices that meets every maximal clique.

* Corresponding author

S. B. Rao & U. K. Sahoo
CR Rao Advanced Institute of Mathematics, Statistics and Computer Science,
Hyderabad, India - 500 046
E-mail: siddanib@yahoo.co.in, umakant.iitkgp@gmail.com

V. Parameswaran
Ministry of Statistics and Programme Implementation, Pune, India - 411 044
E-mail: vp_param@rediffmail.com

He defined stochastic graph to be a vertex labeled graph with non-negative real labels such that every maximal clique has a total weight equal to 1. Parameswaran in his thesis [7], extended this notion to cycles, amongst others. Following this line of research we characterize various kinds of cycle stochastic graphs. We begin by the following definitions.

Edge cycle stochastic: A graph G is said to be edge cycle stochastic (ECS in short) if there is a non-negative real valued function $f_E : E(G) \to \mathbb{R}^+$ such that for every cycle C of G, $f_E(C) = \sum_{e \in E(C)} f_E(e) = 1$.

Vertex cycle stochastic: A graph G is said to be vertex cycle stochastic (VCS in short) if there is a non-negative real valued function $f_V : V(G) \to \mathbb{R}^+$ such that for every cycle C of G, $f_V(C) = \sum_{v \in V(C)} f_V(v) = 1$.

Combining these two ideas it is natural to define the following notion.

Vertex edge cycle stochastic: A graph G is said to be vertex edge cycle stochastic (VECS in short) if there is a non-negative real valued function $f_{VE} : V(G) \cup E(G) \to \mathbb{R}^+$ such that for every cycle C of G, $f_{VE}(C) = \sum_{v \in V(C)} f_{VE}(v) + \sum_{e \in E(C)} f_{VE}(e) = 1$.

For the sake of conciseness, we shall use the short forms VCS, ECS and VECS.

Connections: Due to the close resemblance in the definitions of the above three classes of graphs, it is natural to expect some connections between them.

Remark 1 The following three observations explore the connections between VCS, ECS and VECS graphs.

1. Every VCS graph is ECS. To see this just label edge uv as $f_E(uv) = \frac{1}{2}[f_V(u) + f_V(v)]$.
2. Every ECS (VCS) graph is VECS, since the ECS (VCS) labeling can be extended to VECS labeling by assigning the value 0 to all of its vertices (edges).
3. Every VECS graph is ECS. Define $f_E(uv) = f_{VE}(uv) + \frac{1}{2}[f_{VE}(u) + f_{VE}(v)]$; so for cycle C, $\sum_{v \in V(C)} f_{VE}(v) + \sum_{e \in E(C)} f_{VE}(e) = 1 \Rightarrow \sum f_E(uv) = 1$.

Points 2 and 3 of the above remark prove the following lemma.

Lemma 1 *The class of ECS graphs and VECS graphs are the same.*

So from now onwards we shall be addressing the ECS and VCS graphs only.

Basic Definitions: For the reader's convenience, we recall some basic definitions about graphs which will be useful in this article. A *path* P_k is a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$, for some $k \geq 1$. A *cycle* C_k is a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and edge set of the path P_k including $\{v_kv_1\}$, for some $k \geq 1$. A *chord* of a cycle is an edge joining two of its non-adjacent vertices. A graph, other than a cycle, is said to be *chordless* if none of the cycles in it contain a chord. We exclude the graph to be a cycle in order to avoid conflicts. A *wheel* W_k is formed by joining a vertex to all the vertices of cycle C_{k-1}. This cycle is called the *outer rim* of the wheel. A vertex is said to be a *cut vertex* (or *cutpoint*) c of a graph G, if its removal disconnects G. A graph is said to be *biconnected* if it has no cut vertices. A *block* B of a graph is a maximal biconnected graph. A *block–cutpoint graph* $bc(G)$ of G is defined as the graph with vertex set $\{B_i\} \cup \{c_j\}$ (representing blocks and cut vertices respectively), with two vertices B_i and c_j adjacent if c_j is in B_i. Thus $bc(G)$ is a tree, whose end vertices always represent blocks. An edge is said to be a *cut edge* of a graph G, if its removal disconnects G. A property P is said to be *hereditary* in a class of graphs $\mathcal{G}$, if every induced subgraph of $G \in \mathcal{G}$ also satisfies P. Similarly a property P is said to be *strongly hereditary* in a class of graphs $\mathcal{G}$, if every subgraph of $G \in \mathcal{G}$ also satisfies P. So as the name suggests, *strongly hereditary* is a stronger notion than *hereditary* properties.

By a *contraction* of an edge we mean replacing its end vertices by a new vertex with their adjacencies. A graph G is a *minor* of H if G can be obtained from H by deleting vertices and contracting edges. A *subdivision* of a graph is obtained by adding vertices of degree two into edges. The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G such that adjacent vertices have different colors. CHROMATIC problem deals with determination of the chromatic number of a graph. 3–CHROMATIC and k–CHROMATIC problems deal with checking whether a graph can be colored with 3 and k colors respectively. The *edge chromatic number* $\chi'(G)$ is the smallest number of colors needed to color the edges of G such that edges sharing a common vertex have different colors. The *clique number* $\omega(G)$ is the largest set of pairwise adjacent vertices. CLIQUE problem deals with determination of the clique number of a graph. 3–CLIQUE problem deals with checking whether a graph has clique number greater than equal to 3. A graph of order n is *Hamiltonian* if it contains a cycle of order n. HAMILTONIAN problem deals with checking whether the graph is Hamiltonian. The *feedback vertex set* S of a graph is a set of vertices such that every cycle contains a vertex in S. For other definitions on graph theory (computational complexity), we refer to Harary [5] (Garey and Johnson [3]).

Forbidden Graphs : It should be noted that these classes of graphs are not only hereditary but also strongly hereditary. Due to the folklore result by Hemminger [4] that for any hereditary property there is a set of minimal forbidden graphs. For ECS graphs a forbidden graph characterization was found by Balasubramanian et al. in [1], by means of 5 subdivision classes

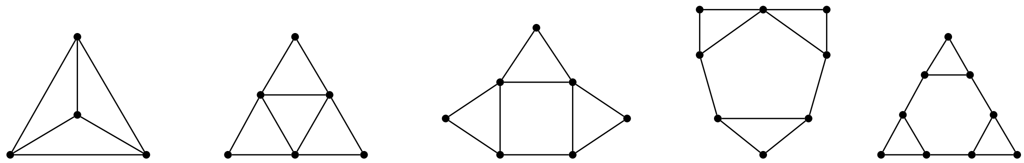


Fig. 1 ECS Forbidden Graphs

shown in Fig 1. We give an alternate proof of this result in Corollary 2. They also proved that a graph is ECS if and only if it has no subgraph contractible to graphs in Fig 1.1 and Fig 1.2. So in addition to the graphs in Fig 1, VCS graphs will have a few more families of forbidden graphs. This is explored later in Section 4.

Objectives and Approach: The main results of this article is outlined in Fig 2. We begin by finding which of the biconnected ECS graphs are also VCS. We use the simple fact that any biconnected graph (block) is either a cycle, or a cycle with a chord, or chordless. This results in a structural characterization of biconnected VCS graphs. We extend this characterization to biconnected ECS graphs. Since in an ECS graph the edges are labeled, joining the blocks (biconnected ECS graphs) at cut vertices preserves the ECS nature of the resultant graph, as no new cycles are created. This gives a structural characterization of ECS graphs. However this is not so trivial in case of VCS graphs because the label of cut vertices should be the same in all of the blocks they are in. So we first found a forbidden graph characterization for biconnected VCS graphs. Using this we found a characterization of VCS graphs. Due to these structural characterizations, it is natural to expect that some of the graph characteristics and problems will be easy to find/solve when restricted to these classes of graphs. Hence we look into various graph characteristics and find that many problems that are NP-complete for general graphs are polynomial for these classes of graphs.

Assumption: As the presence of cut edges does not affect our VCS and ECS labelings, we consider only graphs without cut edges. Every statement regarding computational complexity in this article is under the assumption that $\mathcal{P} \neq \mathcal{NP}$.

Organization: We already have explored the connections between VCS, ECS and VECS graphs in this section. Section 2 deals with the main theme of this paper i.e. development of the structural characterization for biconnected VCS graphs. This is further extended to ECS graphs in Section 3 in form of a structure theorem. In Section 4 we first develop a minimal forbidden subgraph characterization for biconnected VCS graphs and then for VCS graphs; independently using the structural characterization for biconnected VCS graphs (developed in Section 2) and the minimal forbidden graph characterization of VCS graphs we extend the characterization to VCS graphs in Section 5.

$$
\begin{tikzcd}
Graphs \arrow[r, bend left, "structural{[t]}"] \arrow[r, bend right, "forbidden{[t]}"'] & Biconnected\ VCS\ graphs \arrow[r, bend right, "forbidden{[t]}"'] & VCS
\end{tikzcd}
$$

$$
\begin{tikzcd}
Graphs \arrow[r, bend left, "structural{[t]}"] \arrow[r, bend right, "forbidden{[7][t]}"'] & ECS\ graphs
\end{tikzcd}
$$

Fig. 2 Outline of characterization of VCS and ECS graphs. [t] refers to this paper.

We use these structural characterizations to find results on graph characteristics like chromatic number, clique number, edge chromatic number, planarity, Hamiltonianity and length of longest cycle in Section 6. We end with some concluding remarks in Section 7.

2 ECS to VCS

Remark 1.1 proves that every VCS graph is also ECS. Since a characterization of ECS graphs is known, one would naturally be interested in finding when an ECS graph is also a VCS. The following lemma explores such a connection.

Lemma 2 *An ECS graph with no two adjacent vertices with degree greater than* 2 *is always VCS.*

Proof If no two degree 3 vertices are adjacent then the path between such vertices (v_i, v_j) with degree greater than or equal to 3 in cycle C_l contains at least one vertex with degree 2. For all such v_iv_j paths containing no other vertex with degree greater than equal to 3, choose a degree 2 vertex v_k and label it as sum of edge labels of the v_iv_j path. Label all other vertices 0. This results in a VCS. □

Our aim is to find a characterization of VCS graphs. Our approach is as follows: since every graph can be broken down into blocks, we first look into which of these biconnected graphs (blocks) are VCS. Once this is done, and all blocks turn out to be VCS, we try to figure out to appropriately label the vertices of the blocks such that by joining these blocks the VCS is maintained in the whole graph. We begin with the following rudimentary observation.

Lemma 3 *For a set of vertices $S \subset V(G)$ of a biconnected graph G and another vertex $v \notin S$, there are two disjoint paths from v to S.*

Now we tackle the problem of finding when a biconnected graph is VCS; by dividing it into a few sub-problems. If the biconnected graph is a cycle, we are done since it is always VCS. Now we look whether any cycle has a chord.

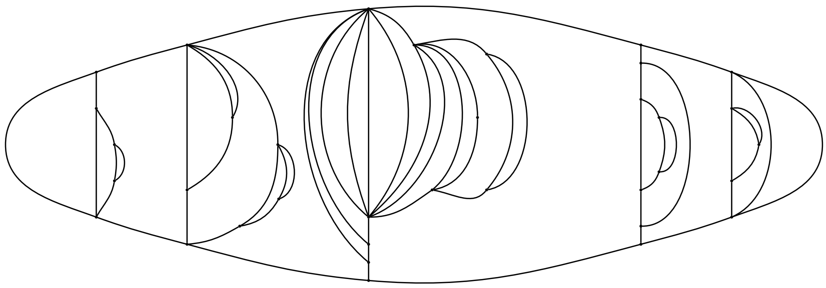


Fig. 3 Cycle with a path i.e. chordless block.

For each edge uv, check for biconnectivity between u and v in $G-uv$. This can be done in linear time [6]. Once such a chord is found we can check whether this graph is VCS using a structural characterization developed later in this article (ref. Lemma 4 and Theorem 1). If no such chords are there, then the block is chordless, and we will have paths instead of chords; for this also we develop a structural characterization to check whether such graph is VCS (ref. Lemma 5 and Theorem 1).

We begin with the following lemma which handles the first non-trivial case where a biconnected graph has a cycle with a chord $\alpha\beta$.

Lemma 4 *A biconnected ECS graph containing a cycle with a chord $\alpha\beta$ is VCS if and only if removal of α or β makes the graph acyclic.*

Proof Clearly $f_V(\alpha)+f_V(\beta)=1$, and rest all vertices in the cycle C have label 0. Also C will have no other chords whose both end points are different from α and β. Now consider a vertex $v \notin V(C)$ (if no such vertex exists, the lemma holds). It has two disjoint paths to $\alpha\beta$. Hence every such vertex has label 0 for G to be a VCS. So α and β are the only vertices with possible non-zero labels since $f_V(\alpha)+f_V(\beta)=1$. If $\alpha\beta$ is the only chord then either all cycles pass through both of them, in which case $f_V(\alpha)$ and $f_V(\beta)$ can have any label respecting $f_V(\alpha)+f_V(\beta)=1$; or all cycles pass through one of them, say α, in which case $f_V(\alpha)=1$. In the first case removal of α or β makes the graph acyclic, where as in the second removal of α makes it acyclic. If there are other chords (and/or paths), then they all must have a common end vertex. Without loss of generality assume it is α, so $f_V(\alpha)=1$, then every cycle in the graph passes through α. So removal of α makes the graph acyclic. □

Remark 2 It is easy to see that whenever $f_V(\alpha)$ is forced to be 1 then the graph is a subdivision of $W_n - e$, where e is an edge in the outer rim of W_n.

Now we handle the case of chordless block i.e cycle with path.

Lemma 5 *A biconnected ECS graph containing no cycle with a chord is VCS if and only if it is of the form as shown in Fig 3.*

Proof This proof is divided into three parts: obstructions, iterative construction and VCS labeling.

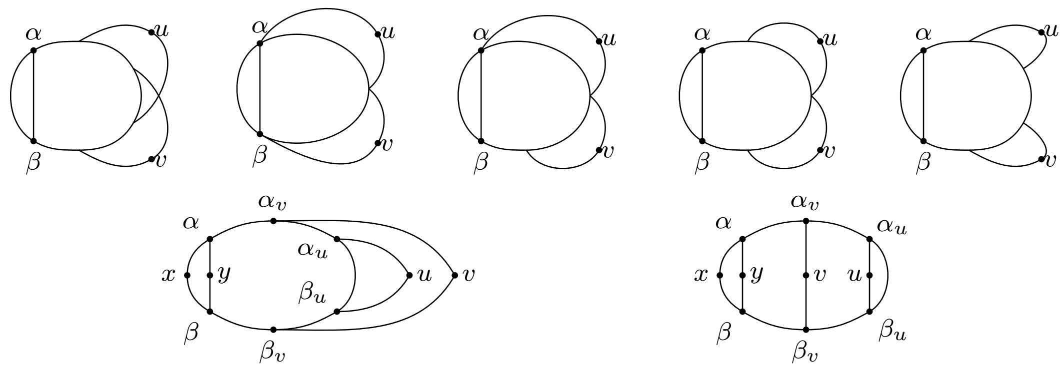


Fig. 4 Obstructions and Allowed configurations

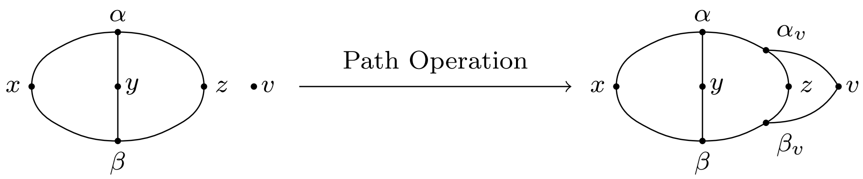


Fig. 5 Path operation

Obstructions: Consider a cycle C with a path $\alpha\beta$ in between, and a vertex v not lying in C or $\alpha\beta$ (if no such vertex exists, the lemma holds). If two disjoint paths from v to $\alpha\beta$ have any vertices common with the cycle C, then they will lie on one side of path $\alpha\beta$, else a K_4–subdivision is induced and ECS is prohibited.

Now consider another vertex u not in the part of the graph formed above. Assume that the two disjoint paths from u to path $\alpha\beta$ lie on the same side of path $\alpha\beta$ as the pair from v. They cannot look like Fig 4.1 to Fig 4.5, else one of the subdivisions of graphs in Fig 1 is induced. Hence the only way this is possible is shown in Fig 4.6. It can be easily checked that the sum of labels in paths $\alpha x\beta$, $\alpha y\beta$, $\alpha_v v\beta_v$, $\alpha_u u\beta_u$ and $\alpha_u\beta_u$ are 1/2 each. (This happens only if labels are positive real. On the same note all labels here belong to $[0, 1]$, as a stochastic function should; $\mathbb{R}^+$ is given in the definitions to maintain consistency with previous articles.)

The pair of disjoint paths from v could have end points in $\alpha\beta$ path without having any common vertices with the cycle. Here it is convenient to notice that any pair of disjoint paths can be treated as another path in the cycle i.e. Fig 4.6 and Fig 4.7 are equivalent. None of these paths are connected in between by chords or paths, else K_4 is induced as a minor. So on iteratively adding the pairs of disjoint paths from all the outer vertices, we end up with something like Fig 3.

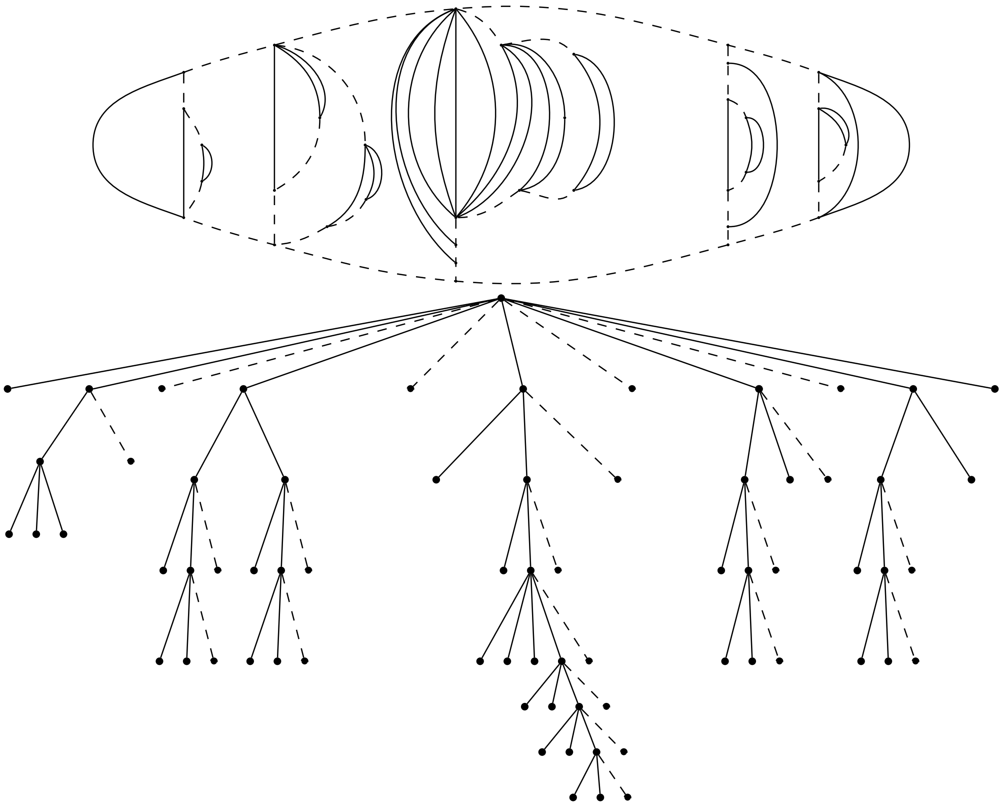
Iterative construction: Now we describe how to construct Fig 3 iteratively. By a *zone* we mean a path or two paths (edges or even vertices) located in a symmetric manner in the cycle with the path $\alpha\beta$. The following description will clear things up. When we initially have a cycle with path $\alpha\beta$, there are three zones i.e. the three $\alpha\beta$ paths: $\alpha x\beta$, $\alpha y\beta$ and $\alpha z\beta$ (ref. Fig 5). On adding two disjoint paths from v we add at most two more zones. Now the zones are $\alpha x\beta$, $\alpha y\beta$, $(\alpha\alpha_v, \beta\beta_v)$, $\alpha_v z\beta_v$ and $\alpha_v v\beta_v$. Please note that if both $\alpha = \alpha_v$ and $\beta = \beta_v$ then only one new zone will be formed, namely $\alpha v\beta$. Two zones are formed even if either $\alpha = \alpha_v$ or $\beta = \beta_v$; in this case the three zones $(\alpha, \beta\beta_v)$, $\alpha z\beta_v$ and $\alpha v\beta_v$ will be replacing $\alpha z\beta$. We also note that there are two types of zones: one containing only one path (eg. $\alpha x\beta$), called as *1–zone*; and the other containing two disconnected components, eg. $(\alpha\alpha_v, \beta\beta_v)$ or $(\alpha, \beta\beta_v)$, called as *2–zone*. By *extremities* of a zone we mean the ordered end vertices of the zone, i.e the extremities of $\alpha x\beta$ are (α, β); the extremities of $(\alpha_v\alpha_u, \beta_v\beta_u)$ are (α_v, β_v) and (α_u, β_u); and the extremities of $(\alpha_v, \beta_v\beta_u)$ are (α_v, β_v) and (α_v, β_u). We define this operation of attaching a new vertex v to the previous graph as a *path* operation (ref. Fig 5). It should be noted that in a path operation, the end points of two disjoint paths belong to the same zone, else K_4 is induced. Furthermore if the end points of these disjoint paths are joined to a 2–zone, then they must join to vertices in different components, else one of the subdivisions of graphs in Fig 1 is induced i.e. for a new vertex w and zone $(\alpha_v\alpha_u, \beta_v\beta_u)$ if one disjoint path from w ends in $\alpha_v\alpha_u$, the other must end in $\beta_v\beta_u$. Hence such paths are added in a nested fashion. We also use a *subdivision* operation which can subdivide any path by adding a new vertex. It is clear that a path operation increases the number of zones usually by 2, or by 1 only when both the endpoints of the two disjoint paths are the extremities of a zone. Now we begin the iterative construction of Fig 3.

Take $K_{2,3}$. Do subdivision and/or path operation. This generates all chordless blocks that are VCS.

VCS labeling: Now we give the VCS labeling. Select a non-extremal vertex in each 1–zone and label it 1/2 and rest all vertices 0. Every possible path from α to β goes through such a vertex with label 1/2. So we have a VCS, and this completes the proof. □

Remark 3 Parallel paths: The number of 1–zones initially added to the cycle (except the 2 on the cycle) are called as *parallel paths* e.g. Fig 3 has five parallel paths and Fig 4.7 has three parallel paths. The number of parallel paths depends on the initial cycle. So any such graph having more that 2 parallel paths can be shown to have just 2 parallel paths by considering two such consecutive paths forming the initial cycle.

Remark 4 Tree structure: The $K_{2,3}$ in the iterative construction contains α and β in the vertex partition with two vertices. It is important to notice that the process of adding 1–zones from $K_{2,3}$ to the final graph can be represented as a tree (ref Fig 6). This is because of the fact that the extremities of a 1–zone to be added belong to the same zone in the graph constructed so far. So each of the new 1–zone is added to only one parallel path (or one of the two 1–zones of the outer cycle), and within each parallel path, this branching is maintained, resulting in a tree structure.



The solid (dashed) lines represent the 1–zones (2–zones). A left-right ordering of 1–zones is maintained in the tree structure. The number of 1–zones and 2–zones in the final graph is same as the respective leaf vertices in the tree.

Fig. 6 Cycle with a path and the tree structure formed by adding 1–zones.

Remark 5 Tree components: It should be noticed that this VCS labeling of giving 1/2 to a non-extremal vertex in a 1–zone works only because every cycle has exactly two 1–zones. In fact every $\alpha\beta$ path goes exactly through one of the 1–zones, and hence contains a vertex with label 1/2. Since there are no paths/chords connecting these $\alpha\beta$ paths in between (else a K_4 is induced), removal of all these 1/2 labeled vertices results in two trees. So if there are t 1–zones i.e. t 1/2–labeled vertices, in a chordless block of order n_b, then there are $n_b + t - 2$ edges in such a block. This also tells us that on removing $t-1$ edges appropriately we have a spanning tree. Such a spanning tree can be got by removing exactly one edge incident to the each of the $t-1$ 1/2–labeled vertices. Because removal of all these 1/2 labeled vertices results in two trees, there is a unique cycle passing through a pair of such vertices, so there are $\binom{t}{2}$ cycles in total. And theses cycles can be found out by joining the two 1/2 labeled vertices with the unique paths in the trees between their neighbours.

Remark 6 Any of the cycles formed in Fig 3 can be treated as the outer cycle. This of course changes the tree structure as described in Remark 4, however the 1–zones and 2–zones remain the same.

The above lemmas prove that a biconnected VCS graph can be labeled with finite labels.

Corollary 1 *Any biconnected VCS graph can have a* $\{0, 1/2, 1\}$*–labeling.*

Proof If the block is a cycle, we can assign label 1/2 to two vertices and 0 to rest, or 1 to a single vertex with 0 to rest. If the block is a cycle with a chord, we just have to make sure that $f_V(\alpha) + f_V(\beta) = 1$. If both α and β have possible non-zero labels i.e. every cycle passes through these two points which means unique chord, then we can assign 1/2 to each (or wlog $(1, 0)$ to (α, β)). However, if there are many chords through α then assign $(1, 0)$ to (α, β). For the third case i.e. cycle with a path, the *VCS labeling* part of the proof of Lemma 5 gives a $\{0, 1/2\}$ labeling. This completes the proof. □

Now we explicitly state the structural characterization of biconnected VCS graphs.

Theorem 1 *A biconnected graph is VCS iff it is a cycle or there exists a vertex whose removal makes it acyclic or is of the form as shown in Fig 3.*

Proof Since the graph is biconnected, it has a cycle. If it is a cycle it is already VCS, else some cycle contains a chord, which can be checked as mentioned earlier in Section 2. In such a case using the proof of Lemma 4 we have such a vertex whose removal makes the block acyclic and this is a VCS. If the block is chordless, then this cycle has a path, say $\alpha\beta$. A similar analysis as in proof of Lemma 5 will result in the construction of Fig 3, and this also is a VCS. □

3 Graphs To ECS

Although we have a forbidden graph characterization of ECS graphs, the techniques used in the proof of Lemma 5 can be used to find its structural characterization, stated as the following theorem.

Theorem 2 *A graph is ECS iff each of its blocks are ECS and are either cycle or of the form as shown in Fig 3 along with some (or no) chords across the cycle parallel to the* $\alpha\beta$ *path.*

Proof If all blocks of a graph are ECS, then joining these blocks at the cut vertices to form the original graph does not add any new cycle, and each edge belongs to exactly one block. Hence the graph is ECS. Now consider each block; since it is biconnected, it has a cycle. If the block is a cycle it is already ECS, else some cycle contains a chord, which can be checked as mentioned earlier in Section 2. We deal with this case later. If the block is chordless, then this cycle has a path, say $\alpha\beta$. A similar analysis as in proof of Lemma 5 will result in the construction of Fig 3; the only difference being that here edges are labeled. So instead of labeling a vertex which is not an extremity in each 1–zone, we label the edge above it as 1/2 and give label 0 to all other edges.

We have a ECS. In the case where cycle has a chord or a path, other parallel chords are allowed (unlike as in VCS). Using a similar analysis as in proof of Lemma 5 and the above labeling, we can have an ECS if we label these chords as 1/2. This concludes our proof. □

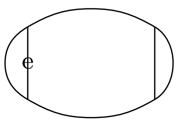
Now we prove the main result of Balasubramanian et al. [1], that the 5 subdivision classes of graphs shown in Fig 1 gives a minimal forbidden graph characterization of ECS graphs. Since a graph is ECS if its blocks are ECS, we consider only biconnected graphs. We call the class of biconnected graphs whose minimal forbidden graphs are the 5 subdivision classes of graphs shown in Fig 1, as $\mathcal{S}-$graphs. Theorem 1 states that all biconnected VCS graphs are $\mathcal{S}-$graphs. The remaining graphs that are not VCS but belong to $\mathcal{S}-$graphs are the last graphs described in Theorem 2, which as proved are ECS. So the $\mathcal{S}-$graphs are precisely the ECS graphs as stated as the following corollary.

Corollary 2 *A graph is ECS if and only if it does not contain any subdivision of graphs shown in Fig. 1.*

Remark 7 Using the above result, it is easy to see that a graph is ECS if and only if it has no subgraph contractible to graphs in Fig 1.1 and Fig 1.2, hence proving the other result of Balasubramanian et al. [1].

4 Minimal Forbidden Graph Characterization for VCS

In order to obtain a similar characterization for VCS graphs, we need to obtain a minimal forbidden graph characterization for VCS graphs. As all VCS graphs are ECS that have a similar minimal forbidden graph characterization as shown in Fig 1, the minimal forbidden graph of VCS contains all these graphs. The following theorem proves that Fig 1 and Fig 7 contains all the minimal forbidden graph for biconnected VCS graphs.



e represents a chord, rest all are paths.

Fig. 7 Biconnected VCS Forbidden graph and No VCS Minor

Theorem 3 *A biconnected graph is VCS iff it has no subgraph isomorphic to the family of graphs given in Fig 1 and Fig 7.*

Proof As stated above, Fig 1 are minimal forbidden graph for VCS also. Now we consider each of the three cases discussed in Section 2 i.e. cycle; cycle with a chord; and chordless block (cycle with a path). Since proofs of these cases in Section 2 excluded the presence of any graphs in Fig 1, and all biconnected graphs fall in these three categories; the forbidden graphs, if any, can be obtained by analyzing these cases only. Every cycle is a VCS, so we concentrate on the later two cases. Also all chordless graphs i.e. containing only cycles with paths, of the form in Fig 3, are VCS. The only restriction, not related to the presence of graphs in Fig 1, comes when the cycle has a chord, where one must ensure that every cycle passes through the end vertex of the chord. So the minimal forbidden graph will violate this restriction. The only minimal forbidden graph of this form is shown in Fig 7. Hence graphs in Fig 1 and Fig 7 are the only minimal forbidden graph for VCS graphs.

□

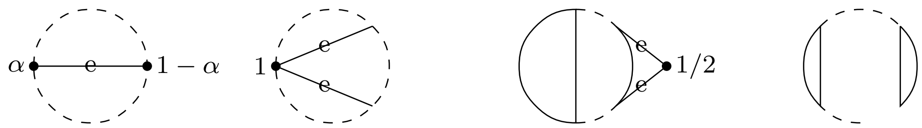
Remark 8 There is no minor characterization of VCS graphs. Fig 7 serves as a counterexample for any such characterization, since it is not VCS where as its subdivisions (where there is no chord) are.

Lemma 6 *Every block in a minimal forbidden graph of VCS graphs has cycles with at most* 2 *parallel paths.*

Proof Assume a minimal forbidden graph of VCS graphs has a block where a cycle has 3 (or more) parallel paths. Clearly this block has a VCS labeling as described in proof of Lemma 5. So there is at least another block attached to it, such that the labels of the cut vertex do not match. Assume these two blocks combined form the minimal forbidden graph. If the cut vertex is in one of the 2–zones, then remove the parallel path farther from it. This results in a smaller graph than the minimal forbidden graph and hence is a VCS. This VCS is still maintained in the previous graph before removing the parallel path. Hence it would not be a minimal forbidden graph. Similarly if the cut vertex is in one of the 1–zones, then removing the parallel path farther from it would not affect its VCS labeling. Hence the result. □

In case of ECS, since the edges are labeled and every edge belongs to an unique block, joining ECS blocks gives a ECS graph. However in case of VCS, problem occurs since a cut vertex can belong to various blocks, and in order to maintain the VCS in the whole graph this cut vertex should have the same label in all these blocks. So this brings in a new set of families of minimal forbidden graph of VCS graphs. Please note that the minimal forbidden graph will have two blocks per cut vertex (it is sufficient to just take the two blocks with different labels). Below we describe how to obtain such minimal forbidden graph.

Description of minimal forbidden graph for VCS: The graphs in Fig 8 represent a set of building graphs, where certain vertices are fixed to have a particular label: 0, 1/2 or 1. These are got from the cases considered while finding the structural characterizations. If the block is a cycle, none of the vertices is forced to have a particular label. In case the block has a cycle with a chord, the vertices of the cycle other than the end points of the chord are forced to have label 0. This gives us the *theta* graph. On the other hand if there are multiple chords, they have to share a common end point which gets label 1, and rest get 0. This results in the *ear* graph. If the block is chordless, then a



(1) Theta Graph (2) Ear Graph (3) 1/2−forced Graph (4) 0−forced Graph

e represents an edge, rest are paths. All vertices on dashed paths have label 0.

Fig. 8 Building Minimal Forbidden Subgraphs for VCS

1/2 label can be forced if a 1–zone has only one non-extremal vertex. The labels of vertices in a 2–zone is always 0. This results in the 1/2–*forced* graph. In case no 1–zones have just one internal vertex, the 2–zones always have vertices with label 0, resulting in a 0–*forced* graph. So in a way a 1/2–forced graph is always a 0–forced graph. Lemma 6 add an upper limit to maximum number of parallel paths. These form the basic building blocks of the set of minimal forbidden graph for VCS. The theta graph has a flexibility in its labeling, its only restriction is the end vertices of the chord should have a total label of 1. Using this we can form a lot of graphs with forced labels, but composed of these blocks e.g. Fig 9.

We know that sum of labels of all vertices in a cycle of a VCS graph is 1; sum of labels of all vertices in a 1–zone of a VCS graph is 1/2; and all vertices in a 2–zone of a VCS graph are labeled 0. These three rules completely cover all the possibilities of obtaining a set of minimal forbidden graph for VCS leading to the following three techniques, each of which result in a set of families (under subdivision) of minimal forbidden graph. For obtaining type–1 minimal forbidden graph, we merge a vertex of a building graph with another vertex of a building graph with a different label. Clearly such graphs are not VCS. We can find a set of minimal forbidden graph from these graphs. For obtaining type–2 minimal forbidden graph, every vertex in a cycle is merged with vertices fixed with label 0 in other building blocks or with at most one fixed vertex label 1/2. Similarly every vertex of a 1–zone is merged with vertex of a building graph with fixed label 0. These operations will result in a set of forbidden graphs from which we can find another set of minimal forbidden graph. For obtaining type–3 minimal forbidden graph, a few (two or three) vertices in a cycle are merged with a vertex of bulding graph with label 1/2 or 1. Similarly at most two non-extremal vertices of a 1–zone is merged with vertices of building graphs with fixed label 1/2 or 1. Care should be taken that sum of these labels is just greater than 1 in case of cycles and 1/2 in case of 1–zones. These operations will result in a set of forbidden graphs from which we can find another set of minimal forbidden graph.

Remark 9 For the purposes of this article, the above description suffices, hence there is no need of explicitly finding the minimal forbidden graph. Furthermore, this list is infinite (ref Fig 9).

Remark 10 In type–1 minimal forbidden graph, a cut vertex is forced to have different labelings in different blocks. In type–2 (type–3) minimal forbidden

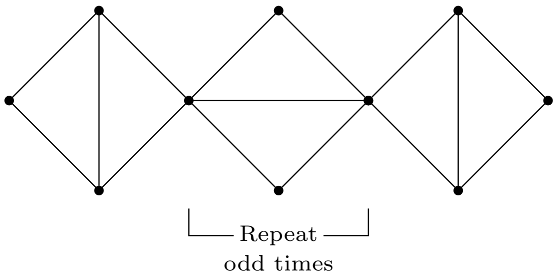


Fig. 9 Infinite Minimal Forbidden Subgraphs for VCS

graph a cycle is forced to have a total labeling less (greater) than 1, or a 1–zone is forced to have a total labeling less (greater) than 1/2.

Now we shall extend Corollary 1 to VCS graphs.

Theorem 4 *Any VCS graph can have a* $\{0, 1/2, 1\}$*–labeling.*

Proof Due to Corollary 1 we have this result for biconnected graphs (blocks). We just have to analyse the cut vertices. Since the graph is VCS, it does not contain any of the three types of the forbidden graphs described above. So if the cut vertex is forced (i.e. there is no other labeling with a different label to it) to have different labels in both blocks, then it contains type–1 minimal forbidden graph. Similarly for every $\{0, 1/2, 1\}$–labeling, if there are type–2 and type–3 minimal forbidden graph then such a can be found in the general labeling, prohibiting it to be VCS. □

5 Graphs To VCS

Using the techniques in the proofs of Lemma 4 and 5, we can find a characterization for VCS graphs — in a similar fashion as in ECS graphs in Section 3. However even if the blocks are VCS, on joining them at cut vertices the VCS property might be lost, as in the case where a cut vertex is forced to have different labels in two of its blocks. So care must be taken while joining such blocks. The minimal forbidden graph for VCS graphs obtained in Section 4 takes care of this. Theorem 1 gives a structural characterization of VCS blocks, and Theorem 3 gives a minimal forbidden graph characterization of VCS blocks. Using these theorems and the minimal forbidden graph developed in Section 4, we have the following theorem.

Theorem 5 *A graph is VCS iff each of its blocks are VCS, and do not contain any of the forbidden graphs described above in Section 4.*

Proof If all blocks of a graph are VCS, then joining these blocks at the cut vertices to form the original graph does not add any new cycle. However it must be ensured that the cut vertices have the same label in all blocks containing that cut vertex. If this occurs then one of the minimal forbidden graph as described in Section 4 occurs, else the graph is a VCS. □

6 Graph Characteristics

In this section we look into various characteristics of VCS and ECS (also VECS) graphs like the chromatic number, clique number, edge chromatic number, Hamiltonianity, length of largest cycle and planarity (also feedback vertex cover for biconnected VCS and ECS graphs).

Lemma 7 *The chromatic number of VCS and ECS (also VECS) graphs is less than or equal to* 3.

Proof The chromatic number of a block which is a cycle or cycle with a chord is less than or equal to 3, equality holds if an odd cycle is present. When the block is a cycle with a path, we can inductively prove that it is 3–colorable due to the presence of a vertex with degree 2. On joining the blocks together appropriate shuffling of the colors can be done because of the block–cutpoint graph is a tree, and hence the chromatic number is preserved. □

Remark 11 The problems 3–CHROMATIC, k–CHROMATIC and CHROMATIC problems are polynomial. To solve CHROMATIC; a graph will have $\chi = 3$ if and only if it does not have $\chi = 1$ or 2. So if there is an odd cycle, $\chi = 3$, else $\chi = 2$ (presence of an edge is obvious).

Lemma 8 *The clique number of VCS and ECS (also VECS) graphs is less than or equal to* 3.

Proof Since K_4 is a forbidden subgraph for ECS graphs (and hence for VCS graphs), the maximum clique that can occur in such graphs is a triangle. This occurs when there is a block which is a triangle or there is a chord between two vertices at distance 2 on a cycle. Furthermore, the clique size remains unaffected on joining the blocks. □

Remark 12 This leads to a polynomial solution to the k–CLIQUE and CLIQUE problems.

Lemma 9 *The edge chromatic number of VCS and ECS (also VECS) graphs is Δ except when it is a odd cycle where it is* 3.

Proof First we deal with VCS graphs. When the block is an even cycle the edge chromatic number χ' is 2 and for an odd cycle it is 3. For a block which is a cycle with a chord, if the chord is unique we already have $\Delta \geq 3$, so we have a Δ–edge coloring. If there are many chords, they must have a common vertex and this vertex has the highest degree, so there exists a $\Delta-$edge coloring. For the case when the block is a chordless graph i.e. cycle with a path, due to Remark 4 we have a ordering of 1–zones that are added to get the block. We prove a Δ–edge coloring by induction. Consider the last 1–zone added. The only problem that appears in obtaining a Δ–edge coloring is when this 1–zone has even number of edges and both the vertices it is going to join to has the maximum degree. Without loss of generality it can be assumed to have 2 edges i.e. the smallest 1–zone. The crucial fact is that every 1–zone has a vertex of degree 2. So on joining this 1–zone to the $\Delta - 1$ degree vertices, it would seem that we might need $\Delta + 1$ colors. But due to the existence of the vertex of degree 2 in the other 1–zone, we can swap one color from one of the $\Delta - 1$ colors with the new Δ^{th} color, in one of the two edges of the 1–zone added, to get a Δ–edge coloring. On joining blocks at a cut vertex if the resultant maximum degree Δ is at the cut vertex we can accordingly introduce new colors in one of the blocks. Since the block–cutpoint graph is a tree, appropriate swapping and renaming the colors of edges will still result in a Δ–edge coloring.

Now for ECS graphs the only difference is the possibility of parallel chords, along with the parallel paths. However in such a case also there exists a vertex with degree 2 in the block. Again we prove by induction. Assume the statement when this vertex is removed. Now on adding this degree 2 vertex, problem occurs if its neighbours have the degree $\Delta - 1$. We also assume that they are connected by a chord, else it is already discussed above. If $\Delta = 3$, we have a 3–edge coloring, else we claim there is at least a path joining these two vertices. This is because if other chords are incident to one of these vertices then the other vertex cannot have degree $\Delta - 1$; so it has to be paths. We will have a degree 2 vertex in such a path. If its neighbours are not the $\Delta - 1$ degree vertices we are done (consider this degree 2 vertex instead of our initial choice). If its neighbours are the $\Delta - 1$ degree vertices we proceed using the argument for the VCS case. Rest of the proof of the VCS part also applies here, resulting in a Δ–edge coloring. □

Lemma 10 *All the VCS and ECS (also VECS) graphs are planar.*

Proof Since all the three building blocks are planar, the resultant VCS and ECS graphs are planar. □

Lemma 11 *The VCS and ECS (also VECS) graphs are Hamiltonian iff they are cycle or cycle with only chords.*

Proof Clearly a cycle and a cycle with only chords are Hamiltonian. Now we rule out the other possibilities. Since the graph has to be biconnected in order to be Hamiltonian we just have to rule out the following: (1) there are paths in cycles with chords and (2) chordless graphs. In both of these cases there is always a path whose internal vertices always stays within a cycle, hence such graphs are not Hamiltonian. □

Lemma 12 *The length of the longest cycle in VCS and ECS (also VECS) graphs can be found in polynomial time.*

Proof Clearly the longest cycle lies in a block, so we have to find the maximum of longest cycles in each block. We consider VCS graphs first. For blocks which are cycle it is trivial. For blocks that are cycles with chords, find the longest path in the tree obtained after removing the common vertex of all cycles (i.e. vertex with maximum degree) add 2 to it. This can be done in polynomial time. In the case where the block is a chordless graph i.e. cycle with a path, the number of cycles is exactly $\binom{t}{2}$ [ref. Remark 5] where t is the number of vertices with label 1/2 in a $\{0, 1/2, 1\}$–labeling i.e. number of 1–zones in the block. Since t is at most $n_b - 2$, the longest cycle in this block can be found in polynomial time. So the problem of determining the length of the longest cycle in a VCS graph is polynomial.

For ECS graphs, the only difference is presence of some parallel chords. We claim that these chords are not part of the longest cycle. If one of these chords is a part of the longest cycle we can always replace it by a longer path, resulting in a longer cycle. Rest of the proof is similar to the VCS part above. So the problem of determining the length of the longest cycle in a ECS graph is also polynomial. □

Remark 13 For a chordless block Remark 5 addresses on how to find these cycles. If (u_i, l_i) are the two neighbours (upper and lower) of a 1/2 labeled vertex v_i then the problem of finding the length of longest cycle in this block can be found by evaluating $4 + \max_{i,j\in[0,t]} d(u_i, u_j) + d(l_i, l_j)$.

Lemma 13 *The feedback vertex cover for biconnected VCS and biconnected ECS (also VECS) graphs can be found in polynomial time.*

Proof The feedback vertex set is the minimum set of vertices S whose removal makes the graph acyclic. As usual we tackle this problem beginning with the chordless blocks. Consider an embedding of the biconnected VCS graph in the plane as shown in Fig 3, and consider its $1/2-$labeled vertices. For each $1/2-$labeled vertex taken left to right, consider the extremities of the 1–zone for the $1/2-$labeled vertex. If both of them do not lie on the outer cycle, then add the extremity with higher degree to S and delete it from the graph. For the rest part of this proof whenever we delete a vertex we also delete all the cut vertices with degree one formed in the process till no such degree one vertex remains. After one left-right scan we have $1/2-$labeled vertices such that at least one of the extremities of its 1–zone is on the outer cycle of the remaining block. Also the original block now might also be broken down into different components, which we can consider iteratively. If only one of the extremity is on the cycle, it must be noted that it will have a higher degree compared to the other extremity. If both extremities are on the outer cycle, then they have the same degree. Now we do another left-right scan. Consider the extremities of each $1/2-$labeled vertex. If only one of the extremity is on the cycle, then add this vertex to S and delete it from the graph. If both extremities are on the outer cycle and their degree is greater than 3, add one of these vertices to S and delete it. After this second left-right scan, we are left with a outer cycle and only parallel paths. In the third scan, add one of each extremity of the fourth 1–zone, sixth 1–zone and so on till the third 1–zone from the right, to S and delete them. Also add one of each extremity of the leftmost and rightmost 1–zone to S. Now S is the feedback vertex cover of this block. In the first two scans, every vertex we added to S belongs to the feedback vertex cover, else at least one cycle will be left out. In the third scan we have to add one from leftmost and rightmost extremities, else the leftmost and rightmost cycle will be left out. For the rest if we add one vertex less to S, then there will be two consecutive 1-zones from which no vertex is in S, so a cycle is left out. This proves the minimality of S. If block is a cycle with chord, or a cycle, removing one vertex suffices. This gives us the minimal feedback vertex cover of biconnected VCS graphs. For biconnected ECS graphs a similar technique can be applied.

7 Conclusion

In this article, we explored various types of cycle stochastic graphs and the connections between them. We gave structure theorems for ECS graphs and biconnected VCS graphs. We also provided an explicit minimal forbidden subgraph characterization for biconnected VCS graphs and then described such a characterization for VCS graphs. Then we use these structural characterization to solve some (usually) hard problems in polynomial time in these graph classes.

One of the aspects of VCS graphs that is not covered here is to find an algorithm to join the VCS blocks to form the VCS graph. As we saw in Section 4, that even if the blocks are VCS the whole graph may not be (e.g. due to the different labels of cut vertices in different blocks). In order to determine whether a VCS graph can be built from VCS blocks, we can make use of Theorem 4, dealing with just three labels. However we suspect, that in a worst case scenario this algorithm would be exponential ($3^{\#\text{cut vertices}} \approx 3^{O(n)}$).

Acknowledgements The authors would like to thank DST CMS GoI Project No. SR/S4/MS: 516/07 dated 21.04.2008 and its extension for their research support.

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