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RANDOM FIXED POINT THEOREMS BASED ON ORBITS OF RANDOM MAPPINGS WITH SOME APPLICATIONS TO RANDOM INTEGRAL EQUATIONS

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Abstract: We obtain some random fixed point theorems for random mappings. We use the orbits of the random mappings to show the existence of a fixed point for a class of random mappings and also establish the measurability of solutions obtained through such random mappings. Some applications of these theorems to random integral equations are given.

Key words: Random operator; Random fixed point theorem; Orbital regularity; Nonexpansive map; Condensing map; Measure of noncompactness; Normal structure; Random integral equaions.

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1 Introduction

Random integral equations and random differential inequalities have applications in modelling physical, engineering , biological, social and system sciences. Random integral equations arise in the study of linear and non-linear differential systems with random parameters. Bharucha-Reid (1972) investigated results such as the existence and the uniqueness of solutions for random linear integral equations such as Fredholm and Volterra integral equations with random forcing functions or with random kernels and random non-linear integral equations of Volterra type wirh random kernels. Ladde and Laksmikantam (1980) studied random differential inequalities and random comparison principles with applications to differential systems

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involving random behavior such as random forcing function or random initial condition or random coefficients. Prakasa Rao and Rama Mohana Rao (1972, 1973) discussed existence and uniqueness of solutions for stochastic integral equations of mixed type among others. Fixed point theorems play a major role in establishing the existence and uniqueness of solutions for integral equations and differentail systems. A survey of fixed point theorems useful for proving the existence and uniqueness of solutions for random integral equation or random differential equations, known as random fixed point theorems in probabilistic frame work, is given in Bharucha-Reid (1976). Prakasa Rao and Rama Mohana Rao (1973) obtained a probabilistic analogue of the Krasnosel'skii's fixed point theorem. Random algebraic equations are investigated in Bharucha-Reid (1970) and random difference equations are studied in Bharucha-Reid (1977). More recent work on random fixed points of completely random operators is presented in Dang Hung Thang and Pham The Anh (2013).

We establish some fixed point theorems for random mappings. We will use orbits to study the existence of a fixed point for a class of random mappings and also establish the measurability of solutions obtained through such random mappings. Some applications of these theorems for random integral equations are given.

2 Preliminaries

We first state few definitions.

Definition 2.1: Let $(\mathcal{X}, \tau)$ be a topological space which is Hausdorff. Let $T : \mathcal{X} \to \mathcal{X}$ be a map. Let $x \in \mathcal{X}$. The set $\{Tx, T^2x, , \ldots\}$ is called the *orbit* of x under the map T. The map T is called *orbitally regular* if it satisfies the following conditions:

$$(R_1): \text{ If } \lim_n T^n x_0 = x_1, \text{ then } \lim_n T(T^n x_0) = Tx_1, \text{ for all } x_0, x_1 \in \mathcal{X}$$

and

$$(R_2): \text{ if } x_0 \neq Tx_0, \text{ then } x_0 \text{ does not belong to the set } \overline{\{T^2x_0, T^3x_0, \ldots\}}$$

where $\overline{A}$ denotes the closure of the set A.

Definition 2.2: Let A be a bounded subset of a metric space (Y, d) with a metric $d(.,.)$. Then the *measure of non-compactness* of the set A (with respect to the space (Y, d)) is defined to be $\gamma_Y(A) = \inf\{\epsilon > 0 :$ The set A can be covered by finitely many subsets of the metric space (Y, d) with diameter $\leq \epsilon\}$.

It can be shown that the function $\gamma_Y(A)$ does not depend on the metric space (Y, d) in the sense that Y and Z are metric spaces and $A \subset Z \subset Y$(the metric on Z being induced by Y), then $\gamma_Y(A) = \gamma_Z(A)$. Here after we write $gamma(A)$ for $\gamma_Y(A)$. The measure $\gamma(A)$ of non-compactness of a set A has the following properties. Suppose A and B are subsets of a metric space Y :

(i) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;
(ii) $\gamma(A \cup B) = \min(\gamma(A), \gamma(B))$;
(iii)$\gamma(A) = \gamma(\bar{A})$;
In addition, suppose Y is a Banach space. Then
(iv) $\gamma(co(A)) = \gamma(A)$ where $co(A)$ denotes the convex hull of the set A;
(v) $\gamma(A + B) = \gamma(A) + \gamma(B)$ where $A + B = \{a + b : a \in A, b \in B\}$;
(vi) $\gamma(\lambda A) = |\lambda|\gamma(A)$ for any $\lambda \in R$ where $\lambda A = \{\lambda a : a \in A\}$. (vii) If the set A is precompact, then $\gamma(A) = 0$.

For proof of these properties, see Nussbaum (1968).

Definition 2.3: Let $(\mathcal{X}, d)$ be a metric space. A map $T : \mathcal{X} \to \mathcal{X}$ is said to be a *condensing* map if T is continuous and if C is any bounded closed subset of $\mathcal{X}$, then $\gamma(T(C)) \leq \gamma(C)$. It is said to be *strictly condensing* if there exists a constant $0 < k < 1$, such that $\gamma(T(C)) < k\,\gamma(C)$ for any bounded closed subset C in the domain of T.

Definition 2.4: Let $(\Omega, \mathcal{B}, P)$ be a complete probability space and $(\mathcal{X}, \zeta)$ be any measurable space. Let $T : \Omega \to \mathcal{X}$ be a function from Ω into $\mathcal{X}$. The function T is called an $\mathcal{X}$-valued *random element* if $T^{-1}E \in \mathcal{B}$ for every $E \in \zeta$.

Definition 2.5: Let $(\mathcal{X}, \zeta)$ and $(\mathcal{Y}, \Sigma)$ be complete normed linear spaces with ζ and Σ the associated Borel σ-algebras under the norm topology. Let T be a map from $\Omega \times D(T)$ into $\mathcal{Y}$ where $D(T) \subset \mathcal{X}$.The function T is called a *random operator* if $\psi(\omega) = T(\omega, x)$ is a $\mathcal{Y}$-valued random element for every $x \in D(T)$.

Definition 2.6: A random operator T is said to be *bounded* if there exists a positive random variable M such that, for every $x \in D(T)$,

$$||T(\omega, x)|| \leq M(\omega)||x|| \text{ a.s. } [P].$$

Definition 2.7: Suppose T is a random operator. The element $\psi(\omega) \in \mathcal{X}$ is called a *random fixed point* of the *random operator* T if $T(\omega, \psi(\omega)) = \psi(\omega)$ a.s. $[P]$ and $\psi(\omega)$ is an χ-valued random element.

We will use the following results on measurable selections and measurability of multi-valued relations or correspondences in the sequel.

Lemma 2.1: *Let $(\Omega, \mathcal{B})$ be a measurable space and S be a complete separable metric space. Suppose that $\Phi : \Omega \to S$. Suppose Φ is a closed-valued correspondence such that the set $\{\omega \in \Omega : \Phi(\omega) \cap F \neq \phi\} \in \mathcal{B}$ for every closed subset F in S. Then there exists a measurable mapping f of Ω into S such that $f(\omega) \in \Phi(\omega)$ for every $\omega \in \Omega$.*

For a proof of this lemma, see Hildenbrand (1974). pp. 55-56.

As a consequence of the Lemma 2.1, the following results can be obtained.

Theorem 2.2: *Let $(\Omega, \mathcal{B}, \mu)$ be a complete measure space and (S, ρ) be a complete separable metric space. Let $\Phi : \Omega \to S$ be a closed-valued correspondence such that for every non-empty closed subset F of S, the set $\Phi^{-1}(F) = \{\omega \in \Omega : \Phi(\omega) \cap F \neq \phi\} \in \mathcal{B}$. Then the correspondence Φ has a measurable selector.*

Proof: Since the set S is a complete separable metric space, every closed subset of S is a G_δ set and hence every open set is a F_σ set. Hence the property that $\Phi^{-1}(F) \in \mathcal{B}$ for every closed subset F implies that $\Phi^{-1}(G) \in \mathcal{B}$ for every open set G in S which is non-empty. The existence of a measurable selector follows from the Kuratowski- Ryll Nardzewski theorem for measurable selectors.

Theorem 2.3: *Let (S, ρ) be a complete separable metric space and $(\Omega, \mathcal{B}, \mu)$ be a complete measure space. Let $\Phi : \Omega \to S$ be a correspondence. Then the following statements are equivalent:*
(i) for every G open in S, the set $\Phi^{-1}(G) = \{\omega \in \Omega : \Phi(\omega) \cap G \neq \phi\}$ belongs to $\mathcal{B}$; and
(ii) the function $\rho(x, \Phi(\omega))$ is a measurable function in ω for every $x \in S$.

Proof: Given $a > 0$, let $B_a(x_0) = \{x : \rho(x_0, x) < a\}$ for any $x_0 \in S$. The map $\omega \to \rho(x, \Phi(\omega))$ is measurable if and only if $\{\omega : \rho(x, \Phi(\omega)) < a\} \in \mathcal{B}$ for any $x \in S$ and $a > 0$. But

$$\{\omega \in \Omega : \rho(x, \Phi(\omega)) < a\} = \{\omega \in \Omega : \Phi(\omega) \cap B_a(x) \neq \phi\} = \Phi^{-1}(B_a(x)) \in \mathcal{B}.$$

Therefore the statements (i) and (ii) are equivalent.

The following result is a consequence of Theorems 2.2 and 2.3 and Lemma 2.1.

Theorem 2.4: *Let (S, ρ) be a complete separable metric space and $(\Omega, \mathcal{B}, \mu)$ be a complete measure space. Suppose the map $\phi : \Omega \to S$ is a correspondence and the space S is σ-compact. Then the set $\Phi^{-1}(C) = \{\omega \in \Omega : C \cap \Phi(\omega) \neq \phi\} \in \mathcal{B}$ for every closed set C in S if and only if the function $\omega \to \rho(x, \Phi(\omega))$ is measurable in ω for every $x \in S$.*

3 Some Fixed Point Theorems

Theorem 3.1: *Let (X, τ) be a metric space. Suppose $T : X \to X$ is a map of X into itself such that T is orbitally regular and there exists at least one point $x_0 \in X$ such that the set*

$$Orb(T, x_0) = \overline{\{Tx_0, \ldots, T^n x_0, \ldots\}}$$

is compact. Then the map T has a fixed point.

Proof: Note that $Orb(T, x) = \overline{\{Tx, T^2x, \ldots\}}$ and $Orb(T, Tx) = \overline{\{T^2x, T^3x, \ldots\}}$ for any $x \in X$. Hence, by the condition (R_1) of orbital regularity, the set $T(Orb(T, x)) \subset Orb(T, x)$ for $x \in X$. Therefore the set $Orb(T, x)$ is invariant under T. For any fixed $x_0 \in X$, let $\mathcal{K}_0$ be the collection of all closed subsets of $Orb(T, x_0)$ invariant under the map T. The set $\mathcal{K}_0$ is non-empty since the set $Orb(T, x_0) \in \mathcal{K}_0$. By the compactness of the set $Orb(T, x_0)$, the intersection of the members of any sub-collection of $\mathcal{K}_0$ is non-empty and it is a non-empty closed subset invariant under T and minimal in the sub-collection partially ordered by inclusion. Therefore, by the Zorn's lemma, there exists a non-empty minimal closed subset of $Orb(T, x_0)$ invariant under T. Let this minimal closed set be K_0. Then K_0 is the closed subset of a compact set and hence K_0 is compact. Suppose $z \in K_0$ and $Tz \neq z$. By the orbital regularity condition (R_2), the element z does not belong to $Orb(T, Tz)$. By invariance, $Orb(T, Tz) \subset K_0$ and z does not belong to $Orb(T, Tz)$. Therefore $Orb(T, Tz)$, is a proper subset of K_0 and is invariant under T. This violates the minimality of K_0. Therefore $Tz = z$ for $z \in K_0$ and $K_0 \neq \phi$. This proves the existence of the fixed point for T.

Theorem 3.2: *Let (X, τ) be a metric space and let $T : X \to X$ be a strict condensing map such that (i) $x \neq Tx$ implies that x is not in $Orb(T, Tx)$ and (ii) there exists x_0 such that $Orb(T, x_0)$ is bounded. Then the map T has a fixed point in $Orb(T, Tx_0)$.*

Proof : Since the map T is condensing, T is continuous. Hence the map T satisfies the condition (R_1) for orbital regularity. The condition (i) stated in the the theorem together with this observation shows that T is orbitally regular. We can establish the existence of the fixed point if we can show that the set $Orb(T, x_0)$ is compact. The set $Orb(T, x_0)$ is closed by its definition and is bounded by the hypothesis. Let $E_0 = Orb(T, x_0)$ and k be the factor of the strict condensing map T where $0 < k < 1$, that is, $k\ \gamma(E_0) > \gamma(TE_0)$ where $\gamma(.)$ is the measure of non-compactness. Suppose $\gamma(E_0) > 0$. Since $TE_0 = E_0$, we have $k\ \gamma(E_0) > \gamma(E_0)$ which is a contradiction since $0 < k < 1$. Therefore $\gamma(E_0) = 0$ and the set E_0 is closed. Therefore $\gamma(E_0) = 0$ implies that E_0 is compact. Hence, by Theorem 3.1, the map T has a fixed point.

Theorem 3.3: *In addition to the conditions of Theorem 3.2, suppose the space (X, τ) is a complete separable metric space. Then there exists a sequence in $Orb(T, x_0)$ which converges to a fixed point of the map T.*

Proof: We have shown earlier that the set $E_0 = Orb(T, x_0)$ is compact under the conditions stated in Theorem 3.2. Hence there exists a convergent subsequence $\{T^{n_i}x_0\} \in E_0$. Since the set E_0 is complete, $\lim_i T^{n_i}x_0 = z_0$ for some $z_0 \in E_0$. By the orbital regularity, it follows that

$$T(\lim_i T^{n_i}x_0) = Tz_0,$$

but

$$T(\lim_i T^{n_i}x_0) = \lim_i T^{n_i+1}x_0 = z_0.$$

Hence $Tz_0 = z_0$ which proves that z_0 is a fixed point of the map T.

Theorem 3.4: *(Darbo (1955)) Let E be a closed bounded set in a Banach space and the map $T : E \to E$ be continuous. Let E_1 be the convex closure of the set $T(E)$, and E_{n+1} be the convex closure of the set $T(E_n)$. Suppose the sequence $\gamma(E_n)$ tends to zero as $n \to \infty$. Then the map T has a fixed point.*

Proof: Since the set E_n is closed, convex and non-empty, it follows that $E_{n+1} \subset E_n$. Let $K = \cap_{n=1}^{\infty} E_n$. Then the set K is non-empty (by a theorem of Kurtowski). Furthermore the set K is closed since each set E_n is closed. Note that $0 \leq \gamma(K) \leq \gamma(E_n)$ for all $n \geq 1$. Since $\gamma(E_n) \to 0$ as $n \to \infty$ by hypothesis, it follows that $\gamma(K) = 0$. Since K is closed, it follows that K is compact. Since the map T is a continuous map from the compact set K into itself, it follows that the map T has a fixed point by the Schauder's theorem.

Theorem 3.5: *Let (X, ρ) be a metric space and $T : X \to X$ be a map such that*
(i) $\rho(Tx, Ty) < \rho(x, y), x \neq y, x, y \in X$;
and
(ii) there exists some $z \in X$ such that $Orb(T, z)$ is compact.
Then the map T has a fixed point.

Proof: Since the map T is a contraction, it is uniformly continuous and the orbital regularity condition (R_1) is satisfied. Suppose $x_0 \neq Tx_0$. Then we have to show that x_0 is not in $Orb(T, Tx_0)$. Since the map T is a contraction, it is obvious that T^m is also a contraction for every $m \geq 1$. Suppose $x_0 = T^m x_0$ for some $m \geq 2$. Then

$$\rho(x_0, Tx_0) = \rho(T^m x_0, TT^m x_0) < \rho(x_0, Tx_0)$$

which is a contradiction. Therefore the assumption that $x_0 = T^m x_0$ for some $m \geq 2$ is not possible. Now suppose that x_0 is a limit point of the set $E_0 = Orb(T, Tx_0)$. Then there exists a sequence $T^{n_i} x_0, i \geq 1$ converging to x_0 and

$$0 \leq \rho(Tx_0, x_0) = \lim_i \rho(T(T^{n_i} x_0), x_0) = 0.$$

Therefore $\rho(Tx_0, x_0) = 0$ which is not possible because $x_0 \neq Tx_0$. Hence x_0 is not in the set $Orb(T, Tx_0)$. Therefore the map T is orbitally regular. By hypothesis, there exists $z \in X$ such that $Orb(T, z)$ is compact. Hence the map T has a fixed point by Theorem 3.1.

4 Random Fixed Point Theorems

We shall now prove some fixed point theorems for random mappings using the fixed point theorems established earlier in Section 3 for non-random mappings.

We shall first prove a theorem for condensing random maps.

Theorem 4.1: *Let $(\Omega, \mathcal{B}, \mu)$ be a complete probability space and X be a separable Banach space. Suppose E is a closed convex subset of X and $T : \Omega \times E \to E$ is a strictly condensing random mapping. Further suppose that the set $\{T(\omega, x) : x \in E\}$ is bounded for every $\omega \in \Omega$. Then the random operator T has a random fixed point.*

We will now prove few lemmas which will be used in the proof of Theorem 4.1.

Lemma 4.2: *Suppose the hypothesis stated in Theorem 4.1 holds. Let $I(\omega) = \{x \in X : T(\omega, x) = x\}$ for any $\omega \in \Omega$. Then the set $I(\omega)$ is closed and compact.*

Proof: We shall first show that the set $I(\omega)$ is non-empty. Let ω be fixed and let $T\omega = T_0$. The operator T_0 is a bounded and strictly condensing map by hypothesis. Let $E_1 = T_0(E)$. Then, by the continuity of the operator T_0, it follows that the set E_1 is closed and it is bounded by hypothesis. Hence the measure of non-compactness of the set E_1, viz.,$\gamma(E_1)$, is defined. Let $E_2 = T_0(E_1)$ and, in general, let $E_n = T_0(E_{n-1})$. Since the operator T_0 is a strictly condensing map, it follows that there exists $0 < k < 1$, depending on ω, such that

$$\gamma(E_n) < k\ \gamma(E_{n-1}), n \geq 2.$$

Hence

$$0 \leq \gamma(E_n) \leq k^{n-1}\ \gamma(E_1)$$

which implies that $\gamma(E_n) \to 0$ as $n \to \infty$. Since the family $\{E_n, n \geq 1\}$ is a decreasing sequence of closed bounded non-empty sets such that $\gamma(E_n) \to 0$, it follows that the set $\cap E_n$ is non-empty and compact by Kuratowski's theorem for complete metric spaces. Let

$$K_0 = \cap_{n=1}^{\infty} E_n.$$

Then the set K_0 is the minimal closed subset invariant under the operator T_0. Furthermore the set K_0 is compact since $\gamma(K_0) = 0$. But the map $T_0 : K_0 \to K_0$ is a continuous map. Hence the operator T_0 has a fixed point which shows that the set $I(\omega)$ is non-empty. We will now prove that the set $I(\omega)$ is closed and compact. Let $\{x_n\}$ be a sequence in $I(\omega)$ such that $x_n \to x_0$ in norm as $n \to \infty$. Then

$$\begin{aligned} ||T_0 x_0 - x_0|| &= ||T\omega x_0 - x_0|| \\ &= ||T\omega x_0 - T\omega x_n + T\omega x_n - x_0|| \\ &\leq ||T_0 x_0 - T_0 x_n|| + ||x_n - x_0|| \end{aligned}$$

since $T(\omega, x_n) = x_n$. From the continuity of the operator $T\omega$, and from the observation that $||x_n - x_0|| \to 0$ as $n \to \infty$, it follows that the last term in the chain of inequalities given above tends to zero as $n \to \infty$ and therefore $||T_0 x_0 - x_0|| = 0$ which proves that $x_0 \in I(\omega)$. Hence the set $I(\omega)$ is closed. We will now show that the set $I(\omega)$ is compact. The set $I(\omega)$ is bounded since $I(\omega)$ is a subset of the the range of the operator $T\omega$. Since the set $I(\omega)$ is bounded and closed, it follows that $\gamma(I(\omega))$ is well-defined. Suppose $\gamma(I(\omega)) = d > 0$. Let $\{x_n\}$ be an arbitrary sequence in $I(\omega)$. The $\gamma(\{x_n\})$ is well defined. Let $\gamma(\{x_n\}) = \rho > 0$. Note that, there exists $0 < k < 1$ such that

$$\gamma(\{T\omega x_n\}) = \rho < k\ \gamma(\{x_n\}) = k\ \rho$$

since the operator $T\omega$ is strictly condensing. Therefore there exists a finite cover $\{S_1, \ldots, S_r\}$ of $\{T_\omega x_n\}$ by closed sets such that, if $T_\omega x_m \in S_i$ and $T_\omega x_n \in S_i$, for any $i = 1, \ldots, r$, then

$$||T_\omega x_n - T_\omega x_m|| < k\rho + \epsilon, \epsilon > 0.$$

Let $K_i = T_\omega^{-1}(S_i), 1 \leq i \leq n$. Then the sets $\{K_1, \ldots, K_n\}$ cover the set $\{x_n\}$ since $T(\omega, x_n) = x_n, n \geq 1$ If x_n and x_m belong to K_i, then $||x_n - x_m|| = ||T_\omega x_m - T_\omega x_n|| < k\rho + \epsilon$. Therefore $\rho < k\rho$. Since $0 < k < 1$, this is not possible. Hence $\rho = 0$. Therefore the closure of the set $\{x_n\}$ is compact. This shows that every arbitrary sequence in $I(\omega)$ has compact closure. Therefore every sequence in $I(\omega)$ has a convergent subsequence and $I(\omega)$ is the closed subset of a complete separable metric space. Hence the set $I(\omega)$ is compact.

Lemma 4.3: *Suppose the hypothesis stated in Theorem 4.1 holds. Let the set E be as stated in Theorem 4.1 and let E be separable. Let $\{y_i\}$ be a countable dense subset of E and F be a non-empty closed subset of E. Let $I^{-1}(F) = \{\omega : I(\omega) \cap F \neq \phi\}$ and $d(x, F) = \inf\{||x - y|| : y \in F\}$. Define*

$$F_n = \{x : d(x, F) < \frac{1}{n}\}.$$

Then

$$I^{-1}(F) = \cap_{n=1}^{\infty} \cup_{y_i \in F_n} \{\omega \in \Omega : ||T_\omega y_i - y_i|| < \frac{2}{n}\}.$$

and $I^{-1}(F) \in \mathcal{B}$.

Proof: Note that $I^{-1}(F) = \{\omega : I(\omega) \cap F \neq \phi\}$ from the definition of the inverse. Let us define

$$M(F) = \cap_{n=1}^{\infty} \cup_{y_i \in F_n} \{\omega \in \Omega : ||T_\omega y_i - y_i|| < \frac{2}{n}\}.$$

Clearly the set $M(F)$ is measurable. It is also easy to see that $\omega \in I^{-1}(F)$ implies that $\omega \in M(F)$ since $\omega \in I^{-1}(F)$ implies that there exists some $x_0 \in F$ such that $T_\omega x_0 = x_0$ and hence, given $n \geq 1$, there exists $y_i \in F_i$ such that

$$||T_\omega y_i - x_0 + x_0 - y_i|| < \frac{1}{n} + \frac{1}{n}.$$

Now we will show that $M(F) \subset I^{-1}(F)$. Let $\omega \in M(F)$. Then there exists a countable subsequence $\{y_{i_k}\}$ of the countable dense sequence $\{y_i\}$ such that

$$d(y_{i_k}, F) < \frac{1}{k}$$

and

$$||T_\omega y_{i_k} - y_{i_k}|| < \frac{1}{k}.$$

Let $A = \{y_{i_k}\}$. Note that

$$\begin{aligned} ||y_{i_k} - y_{i_m}|| &= ||y_{i_k} - T_\omega y_{i_k} + T_\omega y_{i_k} - T_\omega y_{i_m} + T_\omega y_{i_m} - y_{i_m}|| \\ &\leq \frac{2}{k} + \frac{2}{m} + ||T_\omega y_{i_k} - T_\omega y_{i_m}|| \end{aligned}$$

which implies that the measure of non-compactness of the set $\{y_{i_k}\}$ is less than or equal to measure of non-compactness of the set $\{T_\omega y_{i_k}\}$ which is a contradiction if $\gamma(\{y_{i_k}\}) > 0$. Therefore $\gamma(\{y_{i_k}\}) = 0$. Hence the closure of the set A is compact. Therefore there exists a subsequence $\{y_n^*\}$ of $\{y_{i_k}\}$ which converges. Let $\lim_n y_n^* = y_0$. Then $||T_\omega y_0 - y_0|| = 0$ which implies that $T_\omega y_0 = y_0$. Therefore $\omega \in I^{-1}(F)$. Hence $M(F) \subset I^{-1}(F)$. We have shown earlier that the set $M(F)$ contains the set $I^{-1}(F)$. Therefore $M(F) = I^{-1}(F)$. Hence the set

$$I^{-1}(F) = \{\omega : I(\omega) \cap F \neq \phi\} = M(f) = \cap_{n=1}^{\infty} \cup_{y_i \in F_n} \{\omega \in \Omega : ||T_\omega y_i - y_i|| < \frac{2}{n}\} \in \mathcal{B}.$$

Therefore the set $I^{-1}(F)$ is measurable with respect to the measurable space $(\Omega, \mathcal{B})$.

Proof of Theorem 4.1 : Since $I^{-1}(F)$ is measurable for every non-empty closed subset F of E, it follows, by the selection theorem (Lemma 2.1) of Kuratowski- Ryll Nardzewski (cf Hildenbrand (1974)), that there exists a measurable map $f : \Omega \to E$ such that $f(\omega) \in I(\omega)$ and $f(\omega)$ is the required measurable fixed point in $I(\omega)$.

Theorem 4.4: *Let (X, ρ) be a complete separable metric space and E be a closed bounded subset of X. Let $(\Omega, \mathcal{B}, \mu)$ be a complete probability space. Suppose the operator $T : \Omega \times E \to E$ is such that the following conditions are satisfied: (i) T is a random mapping;*
(ii) for every $\omega \in \Omega$, there exists at least one $x_0 \in E$ such that $Orb(T_\omega, x_0)$ is compact; and
(iii) if K_ω is the minimal closed subset of $Orb(T_\omega, x_0)$ which is left invariant by T, then the function $d(x, K_\omega)$ is measurable with respect to $(\Omega, \mathcal{B})$ for all $x \in E$..
Then the operator T has a random fixed point.

Proof : We have already established that, if a map T is an orbitally regular map and if $E_0 = Orb(T, x_0)$ is compact, then E_0 has a minimal closed set which is also compact and invariant under T. Given $\omega \in \Omega$, there exists a set K_ω as defined above which is compact and invariant under T_ω. It is further proved that, for every $z \in K_\omega$, $T_\omega z = z$. Now, consider the mapping

$$I : \Omega \to 2^X$$

such that $I(\omega) = K_\omega$. It is clear that I is a compact-valued correspondence. We shall now establish that $I^{-1}(F) \in \mathcal{B}$ for every non-empty closed subset F of X. This follows by arguments similar to those given for proving $I^{-1}(C) \in \mathcal{B}$ for every non-empty closed subset C in the proof Theorem 4.1. Thus I is a measurable correspondence. Hence, by the Kuratowski-Ryll Nardzewski theorem on selectors, it follows that there exists a measurable map $f : \Omega \to X$ such that $f(\omega) \in I(\omega) = K_\omega$ and hence $T_\omega f(\omega) = f(\omega)$. Hence $f(\omega)$ is the required fixed point of the mapping T_ω.. Note that the measurability of $I^{-1}(F)$ follows from the fact that the function $d(x, I(\omega))$ is measurable and $I(\omega)$ is compact-valued.

We shall now prove some results for families of random mappings.

Definition 4.1: A subset K of a Banach space X has *normal structure* if, for each bounded convex subset H of K which contains more than one element, there exists an element $x \in H$, with the property

$$\sup\{||x - y|| : y \in H\} < \delta(H)$$

where $d\delta(H)$ denotes the diameter of the set H. For any set A contained in the Banach space X, the diameter $\delta(A) = \sup\{||x - y|| : x, y \in A\}$.

Hereafter, we shall denote the diameter of a set A by $\delta(A)$.

Definition 4.2: A mapping $T : X \to X$ is said to be a *non-expansive map* if

$$||Tx - Ty|| \leq ||x - y||, x, y \in X.$$

For any bounded subset H of X, let

$$r_x(H) = \sup\{||x - y||, y \in H\}.$$

For any bounded subset H and any subset K, of X, define

$$r(H, K) = \inf\{r_x(H) : x \in K\}$$

and

$$C(H, K) = \{x \in K : r_x(H) = r(H, K)\}.$$

The set $C(H, K)$ is called the *Chebyshev centre* of H in K.

Lemma 4.5: *If K is weakly compact and convex, and if H is bounded, then the set $C(H, K)$ is a non-empty closed convex subset of K.*

For a proof of Lemma 4.5, see Belluce and Kirk (1967).

Definition 4.3: Let K be a bounded closed convex subset of a Banach space B. The set K is said to have *complete normal structure* if every closed convex subset W contained in K, which contains more than one point, satisfies the following condition (A): for every decreasing net $\{W_\alpha, \alpha \in A\}$ of subsets of W which have the property that $r(W_\alpha, W) = r(W, W), \alpha \in A$, it is the case that the closure of

$$\cup_{\alpha \in A} C(W_\alpha, W)$$

is a non-empty proper subset of W.

The following theorem is due to Belluce and Kirk (1967).

Theorem 4.6: *(Belluce and Kirk (1967)) Suppose K is a weakly compact convex subset of a Banach space B and the set K has a complete normal structure. Let $\mathcal{F}$ be a commutative family of non-expansive mappings f of K into itself. Then there is an element $x \in K$ such that $f(x) = x$ for every $f \in \mathcal{F}$.*

Remarks: Balluce and Kirk (1967) have proved that if a set K is a bounded convex subset of a uniformly convex Banach space, then the set K has complete normal structure.

The next theorem gives sufficient conditions for the existence of a random fixed point for a random operator T which is condensing for every $\omega \in Omega$.

Theorem 4.7: *Let X be a weakly compact convex subset of a separable Banach space E and T be a mapping from $\Omega \times X \to E$ such that $T(\omega, .)$ is a condensing map for every $\omega \in \Omega$.. Suppose that the set E is uniformly convex and for any $\omega \in \Omega$, $T(\omega, x) \in X$ for $x \in X$. Further suppose that the set $\{(T(\omega, x) : x \in E\}$ is bounded for every $\omega \in \Omega$. if $x \in X$, then the mapping T has a random fixed point.*

Proof: Let $x_0 \in X$ and $\{\epsilon_n\}$ be a sequence of positive numbers such that $0 \leq \epsilon_n \leq 1, n \geq 1$ and $\epsilon_n \downarrow 0$ as $n \to \infty$. Define

$$\tilde{T}_n(\omega, x) = \epsilon_n x_0 + (1 - \epsilon_n) T(\omega, x). \tag{4. 1}$$

It can be checked that $\tilde{T}_n(\omega, .)$ is a condensing map with condensation factor $(1-\epsilon_n)k_\omega$ where $0 < k_\omega < 1$, and k_ω is a condensation factor of $T(\omega, .)$. It follows that the map $\tilde{T}_n(\omega, .)$ has a random fixed point by Theorem 4.1. Let $\sigma_n(\omega)$ be the random fixed point of $\tilde{T}_n(\omega, .)$. Define the map I_n from Ω into the family of weak compact subsets of X by the relation

$$I_n(\omega) = \text{weak closure of the set}\{\sigma_n(\omega), \sigma_{n+1}(\omega)......\}.$$

Let $I(\omega) = \cap_n I_n(\omega)$. Note that the weak topology on the space X is the metric topology (cf. Dunford and Schwartz (1958), p 443). Using this observation, we will show that the set I is weakly measurable.This follows from the following result of Himmelberg (1975):

"Let X be a separable and metrizable space and let $F_n : \Omega \to X$ be a weakly measurable relation with closed values for each $n \geq 1$. Further suppose that, for each $\omega \in \Omega$, the set $F_n(\omega)$ is compact for some $n \geq 1$. Then $F = \cap_n F_n$ is measurable."

Applying the result of Himmelberg (1975) stated above, it follows that the set F is weakly measurable. Hence, by the Kuratowski-Ryll Nardzewski theorem, there exists a weakly measurable selector σ of I. For any $x^* \in E^*, x^*(f(.))$ is measurable as a scalar function. Since E is separable and weak-measurability with separability implies measurability, it follows that σ is measurable. For a given $\omega \in \Omega$, there exists some subsequences $\{\sigma_{n_k}(\omega)\}$ of $\{\sigma_n(\omega)\}$ such that $\sigma_{n_k}(\omega)$ converges weakly to $\sigma(\omega)$ as $k \to \infty$. By the uniform convexity of the space E, it follows that the set $I - T(\omega, .)$ is demi-closed by the Browder's theorem (cf. Browder (1965). By the demi-closedness of $I - T(\omega, .) = G(\omega, .)$, since $\{x_n\}$ converges weakly to x_0 and $G(\omega, x_n)$ converges to x_0^*, it follows that $G(\omega, x_0) = x_0^*$. From this property and equation (4.1) , it follows that $\sigma(\omega)$ is a fixed point of $T(\omega, .)$. We have already established that $\sigma(\omega)$ is measurable. Hence $\sigma(\omega)$ is the random fixed point of the random map T.

Theorem 4.8: *Let E be a uniformly convex separable Banach space, K be a closed convex and bounded subset of E. Let $(\Omega, \mathcal{B}, \mu)$ be a complete probability space and $T : \Omega \times K \to K$ be a condensing map for every ω. Suppose that $T(\omega, .)$ is compact for $\omega \in \Omega$. Then there exists $x_0 \in K$ such that $T(\omega, x_0) = x_0, \omega \in \Omega$.*

Proof: The result follows from Theorems 4.1,4.6 and 4.7.

Theorem 4.9: *Let E be a compact convex subset of a separable Banach space X and T :*

$\Omega \times E \to E$ *be a random operator which is compact. Then the operator* T *has a random fixed point.*

Proof: Since the set E is a compact subset of a complete metric space, the set E is bounded. Therefore the set $T(\omega, E)$ is compact for every $\omega \in \Omega$. Therefore, for every ω, the operator $T(\omega, .)$ has a fixed point by the Schauder's fixed point theorem. Let $I : \Omega \to 2^E$ be such that $I(\omega) = \{x : T(\omega, x) = x\}$ as in Lemma 4.2 and Theorem 4.1. The set $I(\omega)$ is closed as $T(\omega, .)$ is continuous. Furthermore $T(\omega, .)$ is a compact operator. Therefore the set $I(\omega)$ is compact. Then, following the arguments similar to those given in Lemma 4.3, it can be shown that the set $I^{-1}(C)$ is measurable for every closed subset C of E. Therefore the operator T has a random fixed point.

Remarks: The condition on the separability of the space X can be relaxed because the space E is compact and hence separable.

Theorem 4.10: *Let* X *be a non-empty bounded closed convex subset of a Banach space* E*. Suppose* M *is a compact convex subset of* X *and* ζ *is a countable family of commuting orbitally regular maps such that, for each* $x \in X$ *and* $f \in \zeta$, $\overline{Orb(f, x)} \cap M \neq \phi$*. Then there exists a point* $x_0 \in M$ *such that* $f(x_0) = x_0$ *for every* $f \in \zeta$*.*

Proof : Let C be any closed non-empty convex subset of X such that $f(C) \subset C$ for every $f \in \zeta$. Choose $x \in C$. Since $f(C) \subset C$, it follows that the $\overline{Orb(f, x)} \subset C$ and therefore

$$C \cap M \supset \overline{Orb(f, x)} \cap M \neq \phi.$$

Therefore $C \cap M \neq \phi$. Let κ be the family of non-empty closed convex subsets of X invariant under ζ. Then $x \in \kappa$ and hence κ is non-empty. Let us order the sets in κ by inclusion. If $K \in \kappa$ and $K \neq \phi$, then $K \cap M \neq \phi$. Every finite subfamily of κ has a non-empty minimal element. By the Zorn's lemma, the family κ has a minimal element K^*. Let $\hat{M} = K^* \cap M$. Then $\hat{M} \neq \phi$. Furthermore, for $x_0 \in \hat{M}$, the set $\overline{Orb(f, x_0)}$ is contained in $\hat{M}$ and the set $\overline{Orb(f, x_0)}$ is a closed subset of M. Hence, by the orbital regularity of the maps f in ζ, each f has a fixed point in $\hat{M}$. By the orbital regularity condition (R_2), it follows that $f(x) = x$ for every $x \in \hat{M}$ and for every $f \in \zeta$.

Theorem 4.11: *Let* X *be a non-empty bounded closed subset of a separable Banach space* E*. Let* $(\Omega, \mathcal{B}, \mu)$ *be a complete probability space. Suppose* Λ *is a countable set and* ω *denote any element* Ω*. Let* $T^\lambda : \Omega \times X \to X$ *be a random map for every* $\lambda \in \Lambda$ *and orbitally regular* *for every $\omega \in \Omega$. Suppose that the family $\{T^\lambda, \lambda \in \Lambda\}$ is a commuting family and the set M is a compact subset of X such that , for every $x \in X$, and for any $\omega \in \Omega$ and $\lambda \in \Lambda$,*

$$\overline{Orb(T^\lambda(\omega, x))} \cap M \neq \phi.$$

Let $F^\lambda(\omega) = \{x : T^\lambda(\omega, x) = x\}$. Then the function $d(x_0, F^\lambda(\omega))$ is a measurable function and the family $\{T^\lambda, \lambda \in \Lambda\}$ has a common random fixed point.

Proof : Given $\omega \in \Omega$, let $F(\omega) = \hat{M}$ where the set $\hat{M}$ is as defined in Theorem 4.10. The set $\hat{M}$ is a compact set and the function $d(x, \hat{M})$ is measurable, since, for each λ, the set $F^\lambda(\omega)$ is such that the function $d(x_0, F^\lambda(\omega))$ is measurable. Therefore the function $F : \Omega \to 2^X$ is itself a measurable multi-function. Now the function F has a measurable selector $\sigma : \Omega \to X$ by Kuratowski-Ryll Nardzewski lemma, $\sigma(\omega) \in \hat{M}$ for every $T^\lambda(\omega, .)$ and $T^\lambda(\omega, \sigma(\omega)) = \sigma(\omega)$ for every λ. Therefore the family $\{T^\lambda(\omega, .), \lambda \in \Lambda\}$ has a common random fixed point.

5 APPLICATIONS

Let C be a closed subset of R and X be the linear space of bounded continuous functions on C. Then the space X is a Banach space under the supremum norm. Let $(\Omega, \mathcal{B}, P)$ be a complete probability space and $k : C \times C \times \Omega \to R^+$ be a continuous bounded function for every $\omega \in \Omega$ generating a ρ-set contraction or strictly condensing map with condensing factor $\rho, 0 < \rho < 1$. Let $f : C \times R^+ \to R^+$ be a bounded continuous function. Let Σ be the cone of nonnegative functions in X. Define a random operator $K : X \to X$ by the mapping

$$Kx(s)(\omega) = \int_C k(s, t, \omega)x(t)dt, \omega \in \Omega$$

for $x \in X$. The operator K is called the *random operator* generated by the function k. Define

$$(5.\ 1) \qquad (Fx)(s) = f(s, x(s)), x \in X.$$

Theorem 5.1: *Let the functions k and f be as defined above. and suppose the following conditions hold*:

(C(i)) *the function $k(s, t, \omega)$ is measurable in ω or every $(s, t) \in C \times C$;*

(C(ii)) $\sup\{\int_C |k(s, t, \omega)|dt, s \in C\} = M < \infty$ a.s. $[P]$;

(C(iii)) *there exists $\lambda > 0$ such that*

$$\inf\{k(s, t, \omega) : s \in C\} > \lambda \ \sup\{k(s, t, \omega) : s \in C\}$$

for every $\omega \in \Omega$ *and for every* $t \in C$;
(C(iv)) *the function* $f(s,t)$ *is a non-decreasing function in* t *for each* $s \in C$ *and is bounded for* $s \in C$ *and* $t \in [0,\infty)$; *and*
(C(v)) there exists $r > 0$, *such that*

$$r - \lambda M \inf\{f(s,r) : s \in C\} < 0$$

and $\delta > 0$ *such that*

$$|f(s,x) - f(s,y)| \leq \delta|x-y|, s \in C, x, y \in [r,\infty).$$

Suppose $0 < \rho\delta < 1$ *where* ρ *is the condensing factor of the condensing map generated by the function* $k(s,t,\omega)$. *Then there exists a non-zero random solution to the random integral equation*

(5. 2) $$x(s,\omega) = \int_C k(s,t,\omega) f(t, x(t,\omega))dt.$$

Proof: Let K be the random operator generated by the function k. Let

$$E = \{x \in X : x(s) \geq 0 \text{ for all } s \in C\}$$

and

$$E' = \{x \in X : \inf\{x(s) : s \in C\} \geq \lambda \sup\{x(s); s \in C\}\}.$$

It can be seen that both the sets E and E' are cones in the Banach space X. It is enough to prove that there is a non-zero measurable solution to the operator equation

$$x = KFx$$

where F is defined by (5.1). Let $T = KF$. We will show that $K(E) \subset E'$ for $\omega \in \Omega$ and the mapping $T(\omega, .)$ is a $\delta\rho$-contraction compressing the cone E'. Observe that $K(E) \subset E$ since $k : C \times C \to R^+$. Suppose $x \in E$. Fix $\omega \in \Omega$. Then

$$\begin{aligned} \inf\{\int_C k(s,t,\omega)x(t)dt : s \in C\} &\geq \int_C \inf\{k(s,t,\omega) : s \in C\}x(t)dt \\ &\geq \lambda \int_C \sup\{k(s,t,\omega) : s \in C\}x(t)dt \text{ by } (C(iii)) \\ &\geq \lambda \sup\{\int_C k(s,t,\omega)x(t)dt : s \in C\} \end{aligned}$$

for every $\omega \in \Omega$. Let $D = \{x \in E' : ||x|| \geq \frac{r}{\lambda}\}$ where r and λ are given by the conditions $(C(iii))$ and $(C(v))$. Note the $x \in E'$ and $||x|| \geq \frac{r}{\lambda}$ imply that $\inf\{x(s) : s \in C\} \geq r$. It follows, from the condition $(C(v))$,that the map $F : D \to E$ is a δ-contraction, Then, by a theorem from Potter (1974), it follows that $T = KF$ is a $\delta\rho$-set contraction. or a strictly condensing map with condensing factor. Suppose $x \in E'$ and $||x|| = \frac{r}{\lambda}$. Then

$$\begin{aligned}(x - Tx)(s) &= x(s) - \int_C k(s,t,\omega)f(t,x(t))dt \\ &\leq ||x|| - \int_C k(s,t,\omega)f(t,\lambda||x||)dt.\end{aligned}$$

The last inequality follows from the definition of the set E' and the condition $(C(v))$. Hence

$$(x - T(x))(s) \leq \frac{r}{\lambda} - \inf\{f(s,r) : s \in C\} \int_C k(s,t,\omega\}dt.$$

Let $s_0 \in C$ such that

$$\int_C k(s_0,t,\omega)dt > M - \epsilon$$

where $\epsilon > 0$ such that

$$r - \lambda(M - \epsilon) \inf\{f(s,r) : s \in C\} < 0$$

following condition $(C(v))$. Then $(x - Tx)(s_0) < 0$ and hence $x - Tx$ is not in E. Since $E' \subset E$, it follows that $x - Tx\ \sigma\ 0$ for all $x \in E', ||x|| = \frac{r}{\lambda}$ where "σ is the ordering induced by E'. Suppose $y \in E'$. Then

$$(y - Ty)(s) = y(s) - \int_C k(s,t,\omega)f(t,y(t))dt \geq \lambda||y|| - MM'$$

by the conditions $(C(ii)), (C(iii))$ and $(C(iv))$ where $M' \geq f(s,y)$ for all $s \in C$ and $y \in (0,\infty)$. Therefore, if $||y|| > \frac{MM'}{\lambda}$, then

$$(y - Ty)\sigma 0$$

for all $y \in E'$ and $||y|| = \gamma$ where γ is a constant greater than $\frac{MM'}{\lambda}$. From the above observations, it follows that $T : F_{\frac{r}{\lambda},\gamma} \to E'$ is a compression of the cone E' for every $\omega \in \Omega$ where

$$F_{\frac{r}{\lambda},\gamma} = \{x \in E' : \frac{r}{\lambda} \leq ||x|| \leq \gamma\}.$$

Therefore the set $\{x \in F_{\frac{r}{\lambda},\gamma} : T_\omega x = x\}$ is nonempty (cf. Potter (1974)). This holds for every $\omega \in \Omega$. An application of Theorem 4.1 implies that there exists a measurable solution of the equation

$$x = Tx, x \in F_{\frac{r}{\lambda},\gamma}$$

for the equation (5.2) which proves the theorem.

Theorem 5.2: *Let $(\Omega, \mathcal{B}, P)$ be a complete probability space, C be a closed subset of R^n and $k(t, s, \omega)$ be a measurable function for every $(s, t), s \in C, t \in C$ as defined in Theorem 5.1. Further suppose that the following conditions hold:*
$(D(i))$ $\sup_{\omega\in\Omega} \sup\{\int_C |k(s, t, \omega)|dt : s \in C\} = M < \infty$;
$(D(ii))$ there exists $\lambda > 0$ such that

$$\inf\{k(s, t, \omega) : s \in C\} > \lambda \ \sup\{k(s, t, \omega) : s \in C\} \text{ for all } \omega \in \Omega, t \in C.$$

$(D(iii))$ *the function $k(s, t, \omega)$ is measurable in ω for any fixed $(s, t) \in C \times C$;*
$(D(iv))$ *the function $f(s, t, \omega)$ is measurable in ω for any fixed $(s, t) \in C \times C$;*
$(D(v))$ *the function $f(s, t, \omega)$ is non-decreasing function in t for any given $s \in C, \omega \in \Omega$ and it is bounded for all $s \in C, t \in [0, \infty), \omega \in \Omega$;*
$(D(vi))$ *there exists $r > 0$ such that*

$$r - \lambda M \inf\{f(s, t, \omega) : s \in C\} < 0, \omega \in \Omega$$

and $\delta > 0$ such that

$$|f(s, x, \omega) - f(s, y, \omega)| \leq \delta \ |x - y|, s \in C, \omega \in \Omega, r \leq x, y < \infty.$$

$(D(vii))$ *Suppose $0 < \rho\delta < 1$ where ρ is the condensing factor of the condensing map generated by the function $k(s, t, \omega)$.*

Then there exists a a measurable solution to the stochastic integral equation

$$x(t, \omega) = \int_C k(t, s, \omega) f(s, x(s, \omega), \omega) ds, \omega \in \Omega. \tag{5. 3}$$

Proof : The proof is similar to that given for proving Theorem 5.1. An application of Theorem 4.1 shows the existence of a random solution to the equation (5.3).

Let $(\Omega, \mathcal{B}, P)$ be a complete probability space and μ be a complete σ-finite measure on $(R^+, \mathcal{B}_1)$ where $\mathcal{B}_1$ is the σ-algebra of Borel subsets on $(0, \infty)$. Let $x(t, \omega), h(t, x(t, \omega))$ and $f(t, x(t, \omega), \omega)$ be n-dimensional random vectors for each $t \in R^+$. Let $\Sigma^{(n)}$ denote the linear space spanned by the real-valued n-dimensional random vectors defined on the probability space $(\Omega, \mathcal{B}, P)$ such that each component of the random vector belongs to $L_\infty(\Omega, \mathcal{B}, P)$. For any $x(\omega) = (x_1(\omega), \ldots, x_n(\omega)) \in \Sigma^{(n)}$, define

$$||x||_{\Sigma^{(n)}} = \max_{1 \le k \le n} \{ess - \sup_{\omega \in \Omega} |x_k(\omega)|\}.$$

with the term $ess - \sup$ is taken with respect to the probability measure P. Let $g : R^+ \to R^+$ be a real-valued positive continuous strictly increasing function such that $g(t) \to \infty$ as $t \to \infty$. Define the S_g-norm of a random vector $x(t, \omega)$ by

$$||x(t, \omega)||_{S_g} = \sup_{t \in R^+} \{\frac{||x(t, \omega||_{\Sigma^{(n)}}}{g(t)}\}.$$

Let S_0 be the set of continuous mappings from R into $\Sigma^{(n)}$. Further suppose that, if $x \in S_0$, then the function $f(s, x(s, \omega), \omega)$ is continuous from R into $\Sigma^{(n)}$. In addition, suppose that the function $k(t, s, .) : R^+ \times R^+ \to L_\infty(\Omega, \mathcal{B}, P)$ is continuous in t for almost all $s \in R^+$. Furthermore, for every $t \in R^+$ and $x \in S_0$,

$$\int_{R^+} ||k(t, s, \omega)x(s, \omega)||_{\Sigma^{(n)}} d\mu(s) < \infty.$$

Further suppose that there exists a function Λ defined μ-a.e such that

$$||(k(t, u, \omega) - k(s, u, \omega))x(u, \omega)||_{\Sigma^{(n)}} < \Lambda(u)||x(u, \omega)||_{\Sigma^{(n)}} \mu - a.e.$$

and

$$\Lambda(u)||x(s, \omega)||_{\Sigma^{(n)}} \text{ is } \mu - \text{ integrable for } x \in S_0.$$

The conditions stated above ensure that the integral

$$\int_C k(s, t, \omega)x(s, \omega)d\mu(s)$$

exists. Let S_g be the set of continuous functions $x(t, \omega)$ from R into $\Sigma^{(n)}$ such that $||x(t, \omega)||_{S_g} < \infty$.

Theorem 5.3: *Suppose the conditions stated above hold with the function. Let C be a closed subset of R^n. Further suppose that the following conditions hold with $g(t) = e^{\tau t}$ for some $\tau > o$:*

$(E(i))$ *there exists constants $\theta > 0$ and $\alpha > \tau$ such that*

$$\int_C ||k(s, t, \omega)||g(s)d\mu(s) < \theta\, e^{-\alpha t}, t \in R^+;$$

$(E(ii))$ *the function* $f(t, x(t,\omega), \omega)$ *is a continuous function from* R^+ *into the linear space* $\Sigma^{(n)}$ *such that,*
for $||x(t,\omega)||_{S_g} \leq \rho, ||y(t,\omega)||_{S_g} \leq \rho,$

$$||f(t, x(t,\omega), \omega) - f(t, y(t,\omega), \omega)||_{\Sigma^{(n)}} < \lambda\, ||x(t,\omega) - y(t,\omega)||_{\Sigma^{(n)}, t \in R^+}$$

where λ *and* ρ *are positive constants.*
$(E(iii))$ *the function* $h(t, x(t,\omega))$ *is a continuous function from* R^+ *into* $\Sigma^{(n)}$ *such that for* $||x(t,\omega)||_{S_g} \leq \rho, ||y(t,\omega)||_{S_g} \leq \rho,$

$$||h(t, x(t,\omega)) - h(t, y(t,\omega))||_{\Sigma^{(n)}} < \gamma ||x(t,\omega) - y(t,\omega)||_{\Sigma^{(n)}, t \in R^+}$$

where γ *and* ρ *are positive constants. Then there exists a unique random solution to the random integral equation*

$$x(t,\omega) = h(t, x(t,\omega)) + \int_C k(s,t,\omega) f(s, x(s,\omega), \omega) d\mu(s), \omega \in \Omega. \quad (5.\ 4)$$

We shall prove a lemma before proving the Theorem 5.3.

Lemma 5.4: *Let the kernel* $k(s,t,\omega)$ *be defined as given above. Let* S_0 *be the set of all continuous functions from* R^+ *into* Σ^n*. Define*

$$(\Gamma x)(t,\omega) = \int_C k(s,t,\omega) x(s,\omega) d\mu(s), x \in \Sigma^{(n)}.$$

Then the function Γ *maps* S_0 *into itself and the mapping is continuous.*

Proof: Note that the set S_0 is a locally convex space with the topology of uniform convergence on compact sets and it is a Frechet space with the metric

$$\rho(x, y) = \Sigma_{m=1}^{\infty} \frac{1}{2^m} \frac{||x - y||_m}{1 + ||x - y||_m}$$

where

$$||x(t,\omega)||_m = \sup_{t \in K_m} ||x(t,\omega)||_{\Sigma^{(n)}}, K_m = [0, m], m \geq 1.$$

Note that $R^+ = \cup_{m=1}^{\infty} K_m$. Let

$$(\Gamma_m x)(t,\omega) = \int_{K_m} k(t,s,\omega) x(s,\omega) d\mu(s).$$

Suppose $x \in S_0$. Let $t_n \to t$ as $n \to \infty$. It is easy to see that

$$||(\Gamma x)(t_n, \omega) - (\Gamma x)(t, \omega)||_{\Sigma^{(n)}} \to 0$$

as $n \to \infty$ by the dominated convergence theorem. Hence $\Gamma x \in S_0$ if $x \in S_0$. Suppose that $x_j \to x$ in S_0, that is, the sequence of functions $\{x_j(t, \omega)\}$ converge to the function $x(t, \omega)$ in the space S_0 endowed with the topology of uniform convergence on compact sets.Let $n \geq 1$. Then

$$\begin{aligned} ||(\Gamma_m x_j)(t_n, \omega) - (\Gamma_m x)(t, \omega)||_n &= \sup_{t \in K_n} || \int_{K_m} k(t, s, \omega) x_j(s, \omega) d\mu(s) \\ &\quad - \int_{K_m} k(t, s, \omega) x(s, \omega) d\mu(s)||_{\Sigma^{(n)}} \\ &\leq \sup_{t \in K_n} \int_{K_m} ||k(t, s, \omega)|| ||x_j(s, \omega) - x(s, \omega)||_{\Sigma^{(n)}} d\mu(s). \end{aligned}$$

Note that $||x_j(s, \omega) - x(s, \omega)||_{\Sigma^{(n)}} \to 0$ as $j \to \infty$ uniformly on the compact set K_m. Hence, given $\epsilon > 0$, there exists n_0 depending on m such that, for every $j > n_0$,

$$||(\Gamma_m x_j)(t, \omega) - (\Gamma_m x)(t, \omega)||_n \leq \epsilon \sup_{t \in K_n} \int_{K_m} ||k(t, s, \omega)|| d\mu(s)$$

by the definition of the operator Γ_m. Observe that the function

$$\int_{K_m} ||k(t, s, \omega)|| d\mu(s)$$

is a continuous function in $t \in K_n$ and the set K_n is compact. Therefore there exists a constant $B_n > 0$ such that

$$\int_{K_m} ||k(t, s, \omega)|| d\mu(s) < B_n, t \in K_n.$$

Hence, for $j > n_0$,

$$||(\Gamma_m x_j)(t, \omega) - (\Gamma_m x)(t, \omega)||_n \leq B_n \epsilon.$$

Hence, for each $m \geq 1, (\Gamma_m x_j)(t, \omega)$ converges to $(\Gamma_m x)(t, \omega)$ as $j \to \infty$. Furthermore $(\Gamma_m x)(t, \omega)$ converges to $(\Gamma x)(t, \omega)$ as $m \to \infty$. Hence, by a theorem in Dunford and Schwartz (1958), p.54, it follows that Γ is continuous on S_0.

Proof of Theorem 5.3: Let

$$(Qx)(t, \omega) = h(t, x(t, \omega)) + \int_C k(t, s, \omega) f(s, x(s, \omega), \omega) d\mu(s)$$

where $Q : E \to E$ and $E = \{x : x \in S_g \text{ and } ||x||_{S_g} \leq \rho\}$. We will show that the operator Q is a random contraction mapping. Observe that, for $x \in S_g$,

$$
\begin{aligned}
||(\Gamma x)(t,\omega)||_{\Sigma^{(n)}} &\leq \int_C ||k(t,s,\omega)||||x(s,\omega)||_{\Sigma^{(n)}} d\mu(s) \\
&< ||x(t,\omega)||_{S_g} \int_C ||k(s,t,\omega)||g(s)d\mu(s) \\
&< ||x(t,\omega)||_{S_g} \theta e^{-\alpha t} < \infty.
\end{aligned}
$$

Hence $||\Gamma x||_{S_g} < \infty$ and $(\Gamma x)(t,\omega) \in S_g$. Therefore $\Gamma(S_g) \subset S_g$. By Lemma 5.4 and the closed graph theorem, it follows that Γ is a bounded linear operator on S_g. Note that

$$||(Qx)(t,\omega)||_{S_g} \leq ||h(t,x(t,\omega))||_{S_g} + \theta e^{-\alpha t}||f(t,x(t,\omega),\omega)||_{S_g}.$$

Let $\beta > 0$ such that $\beta > \theta$ and $\gamma + \lambda\beta < 1$. Then

$$||(Qx)(t,\omega)||_{S_g} \leq (\gamma + \lambda\beta)||x(t,\omega)||_{S_g} + ||h(t,0)||_{S_g} + \beta||f(t,0,\omega)||_{S_g} \leq \rho.$$

Therefore $(Qx)(t,\omega) \in E$ whenever $x(t,\omega) \in E$. Similarly, for $x \in E$ and $y \in E$,

$$||(Qx)(t,\omega) - (Qy)(t,\omega)||_{S_g} \leq (\gamma + \lambda\beta)||x(t,\omega) - y(t,\omega)||_{S_g}.$$

Therefore the mapping $Q : E \to E$ is a contraction mapping. By the existence of a random fixed point for non-linear random contraction mapping (cf. Bharucha-Reid (1976)), the equation (5.4) has a solution which is an almost surely continuous vector-valued process. Suppose there exist two solutions x and y. Observe that

$$||x(t,\omega) - y(t,\omega)||_{\Sigma^{(n)}} \leq ae^{-bt}$$

for some $a > 0$ and $b > 0$ and the last term tends to zero as $t \to \infty$ almost surely. Hence the solution is unique almost surely.

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