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MORE ON MAXIMAL INEQUALITIES FOR DEMISUBMARTINGALES USING PATHWISE APPROACH

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ABSTRACT: Maximal inequalities for demimartingales and demisubmartingales have been derived earlier using the upcrossing inequalities for demisubmartingales. An alternate approach for deriving maximal inequalities for nonnegative demisubmartingales, using elementary inequalities for real numbers, was given in Prakasa Rao (*Statist. Probab. Lett.*, **82**, (2012) 1388-1390) following Acciaio et al. (*Ann. Appl. Probab.*, **23** (2013), 1494-1505). We now derive improved maximal inequalities for demisubmartingales using a pathwise approach following Gushchin (arXiv:1410.8264v1 [mathPR] 30 Oct 2014).

Key words : Maximal inequalities; Demimartingales; Demisubmartingales; Pathwise approach.
MSC: 60G48.

1 Introduction

In a recent paper, Acciaio et al. (2013) presented a unified approach to derive Doob's L^p maximal inequalities for nonnegative submartingales for $1 \leq p < \infty$. They derive the inequalities as consequences of some elementary inequalities for sequences of real numbers. We used the same technique for obtaining maximal inequalities for nonnegative demisubmartingales in Prakasa Rao (2012b). An extensive discussion on demimartingales, demisubmartingales and their properties is given in Prakasa Rao (2012a). Recently Guschin (2014) presented pathwise counterparts of Doob's maximal inequalities on the probability of exceeding a level. Substituting a trajectory of a stochastic process in his inequalities, he obtained Doob's inequalities for supermartingales and submartingales. He has also derived the pathwise counterpart of Doob's maximal L^p- and $L \log L$-inequalities using this approach. We will now derive similar inequalities for demimartingales and demisubmartingales applying some inequalities for real numbers due to Gushchin (2014).

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Recall that a sequence of random variables $\{S_n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a *demimartingale* if, for every componentwise nondecreasing function g,

$$E[(S_{j+1} - S_j)g(S_1, \ldots, S_j)] \geq 0, j \geq 1$$

assuming that the expectation exists. If the function g is required to be nonnegative and nondecreasing componentwise, then the process $\{S_n, n \geq 1\}$ is called a *demisubmartingale*. If the function g is required to be non-positive and nondecreasing componentwise, then the process $\{S_n, n \geq 1\}$ is called a *demisupermartingale*.

For some examples of demimartingales and demisubmartingales and study of their properties, see Prakasa Rao (2012a). As has been pointed out in Prakasa Rao (2012a), a square integrable submartingale with the natural filtration is a demisubmartingale but there are demisubmartingales which are not submartingales. Hence the results derived here cover a strictly larger class of processes than the class of square integrable submartingales with natural filtration.

2 Main Results

For any finite sequence of real numbers $x_0, \ldots, x_n$, define $x_n^{\max} = \max\{x_0, \ldots, x_n\}$. Let I_A denote the indicator function of a set A. The following theorem is due to Gushchin (2014).

Theorem 2.1 : *Let $x_0, \ldots x_n$ be real numbers. Then, for any $\lambda \in R$,*

$$\lambda\, I_{[x_n^{\max} \geq \lambda]} \leq \min(x_0, \lambda) + \sum_{k=1}^{n} I_{[x_{k-1}^{\max} < \lambda]}(x_k - x_{k-1}) - x_n I_{[x_n^{\max} < \lambda]} \tag{2. 1}$$

and

$$\lambda\, I_{[x_n^{\max} \geq \lambda]} \leq -(x_0 - \lambda) I_{[x_0 \geq \lambda]} - \sum_{k=1}^{n} I_{[x_{k-1}^{\max} \geq \lambda]}(x_k - x_{k-1}) + x_n I_{[x_n^{\max} \geq \lambda]}. \tag{2. 2}$$

Suppose the sequence $\{S_j, 0 \leq j \leq n\}$ is a demisubmartingale. Let

$$g(x_0, \ldots, x_j) = I_{[x_j^{\max} \geq \lambda]}.$$

Note that the function g is nonnegative and nondecreasing componentwise. Hence, by the property of a demisubmartingale, it follows that

$$E[(S_{j+1} - S_j)g(S_0, \ldots, S_j)] \geq 0, 0 \leq j \leq (n-1). \tag{2. 3}$$

Therefore

$$(2.\ 4)\qquad E((S_{j+1}-S_j)I_{[S_j^{\max}\geq\lambda]})\geq 0, 0\leq j\leq (n-1).$$

Applying the second part of Theorem 2.1 to the sequence $\{S_0(\omega),\ldots,S_n(\omega)\}$ and taking expectations on both sides of the inequality (2.2), we get that

$$(2.\ 5)\qquad \lambda\, P(S_n^{\max}\geq\lambda)$$
$$\leq -E[(S_0-\lambda)I_{[S_0\geq\lambda]}]+\int_{[S_n^{\max}\geq\lambda]}S_n dP-\sum_{k=1}^{n}E[I_{[S_{k-1}^{\max}\geq\lambda]}(S_k-S_{k-1})].$$

In view of the inequality (2.4), it follows that

$$(2.\ 6)\qquad \lambda P(S_n^{\max}\geq\lambda)\leq -E[(S_0-\lambda)I_{[S_0\geq\lambda]}+\int_{[S_n^{\max}\geq\lambda]}S_n dP.$$

Suppose the sequence $\{S_j, 0\leq j\leq n\}$ is a demisupermartingale. Let

$$h(x_0,\ldots,x_j)=-I_{[x_j^{\max}<\lambda]}.$$

Note that the function h is nonpositive and nondecreasing componentwise. Hence, by the property of a demisupermartingale, it follows that

$$(2.\ 7)\qquad E[(S_{j+1}-S_j)h(S_0,\ldots,S_j)]\geq 0, 0\leq j\leq (n-1).$$

Therefore

$$(2.\ 8)\qquad E[(S_{j+1}-S_j)I_{[S_j^{\max}<\lambda]}]\leq 0, 0\leq j\leq (n-1).$$

Applying the first part of Theorem 2.1 to the sequence $\{S_0(\omega),\ldots,S_n(\omega)\}$ and taking expectations on both sides of the inequality (2.1), we get that

$$(2.\ 9)\ \lambda\, P(S_n^{\max}\geq\lambda)\leq E[\min(S_0,\lambda)]-\int_{[S_n^{\max}<\lambda]}S_n dP+E[\sum_{k=1}^{n}I_{[S_{k-1}^{\max}<\lambda]}(S_k-S_{k-1})].$$

In view of the inequality (2.8), we have the following inequality for demisupermartingales: for any $\lambda\in R$,

$$(2.\ 10)\qquad \lambda\, P(S_n^{\max}\geq\lambda)\leq E[\min(S_0,\lambda)]-\int_{[S_n^{\max}<\lambda]}S_n dP.$$

Combining the inequalities (2.6) and (2.10), we have the following main result.

Corollary 2.2: *Let $\lambda \in R$. Then*
(i) if $\{S_k, 0 \leq k \leq n\}$ is a demisupermartingale, then

$$\lambda\, P(S_n^{\max} \geq \lambda) \leq E[\min(S_0, \lambda)] - \int_{[S_n^{\max} < \lambda]} S_n dP \tag{2. 11}$$

and
(ii) if $\{S_k, 0 \leq k \leq n\}$ is a demisubmartingale, then

$$\lambda\, P(S_n^{\max} \geq \lambda) \leq -E[(S_0 - \lambda) I_{[S_0 \geq \lambda]} + \int_{[S_n^{\max} \geq \lambda]} S_n dP. \tag{2. 12}$$

The inequality (2.12) derived above for demisubmartingales is an improvement over the corresponding inequality (2.7.1) in Prakasa Rao (2012a), p.54.

The next lemma was proved in Acciaio et al. (2032) and it follows also from the second part in Theorem 2.1 stated above due to Gushchin (2014).

Theorem 2.3: *Let $x_0, \ldots, x_n$ be nonnegative real numbers. Let $p > 1$ and $q = \frac{p}{p-1}$. Then*

$$[x_n^{\max}]^p \leq q^p x_n^p - q x_0^p - qp \sum_{k=1}^{n} [x_{k-1}^{\max}]^{p-1} [x_k - x_{k-1}]. \tag{2. 13}$$

Let the sequence $\{S_k, 0 \leq k \leq n\}$ be a nonnegative demisubmartingale such that $E(S_n^p) < \infty$. Let

$$v(x_1, \ldots, x_j) = [x_j^{\max}]^{p-1}.$$

Then the function v is a nonnegative function and nondecreasing componentwise. Hence

$$E[(S_{j+1} - S_j) v(S_0, \ldots, S_j)] \geq 0, 0 \leq j \leq (n-1). \tag{2. 14}$$

Applying Theorem 2.3 pathwise, taking expectations on both sides of the inequalities so derived and using the inequality (2.14) , we obtain the following result.

Corollary 2.4: *Let $\lambda \in R$. Suppose that the sequence $\{S_k, 0 \leq k \leq n\}$ is a nonnegative demisubmartingale with $E|S_n^p| < \infty$. Then*

$$E([S_n^{\max}]^p) \leq q^p E[S_n^p] - q E[S_0^p]. \tag{2. 15}$$

This result is analogue of a corresponding result of Cox (1984) for nonnegative submartingales. Choosing $p = 2$ in the Corollary 2.4, we get that

$$E([S_n^{\max}]^2) \leq 4E[S_n^2] - 2E[S_0^2]. \tag{2. 16}$$

for any nonnegative demisubmartingale $\{S_k, 0 \leq k \leq n\}$.

The following theorem is proved in Acciaio (2013) and an alternate proof of was given in Gushchin (2014).

Theorem 2.5: *Suppose $x_0 > 0$ and $x_1, \ldots, x_n$ are nonnegative. Then*

$$x_n^{\max} \leq \frac{e}{e-1}[x_0 + x_n \log(x_n/x_0) - \sum_{k=1}^{n} \log(x_{k-1}^{\max}/x_0)(x_k - x_{k-1})]. \tag{2. 17}$$

Let

$$w(x_1, \ldots, x_j) = \log[x_j^{\max}/x_0].$$

Then the function w is nondecreasing componentwise. Let the sequence $\{S_k, 0 \leq k \leq n\}$ be a nonnegative demimartingale. Then

$$E[(S_{j+1} - S_j)w(S_0, \ldots, S_j)] \geq 0, 0 \leq j \leq (n-1). \tag{2. 18}$$

Applying Theorem 2.5, we get the following corollary by arguments similar to those given above.

Corollary 2.6: *Let $S_0 > 0$ a.s. and the sequence $\{S_k, 0 \leq k \leq n\}$ be a nonnegative demimartingale. Then*

$$E([S_n^{\max}) \leq \frac{e}{e-1}[E(S_0) + E(S_n \log(S_n/S_0))]. \tag{2. 19}$$

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