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Complexity Results in Self-Complementary Graphs[*]

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Abstract

A *self-complementary* (di)graph is a (di)graph isomorphic to its complement. In this paper we prove that, for a self-complementary graph G, finding its clique number $\omega(G)$, independence number $\alpha(G)$ and chromatic number $\chi(G)$ are all $\mathcal{NP}$-hard, whereas checking whether its chromatic number is less than or equal to an integer k, for $k \geq 3$; and whether its clique number and independence number are greater than or equal to an integer k, for $k \geq 3$, are in $\mathcal{P}$. We further prove that the domination number $\gamma(G)$ and the differences $\chi(G) - \omega(G)$, $\chi(G) - \gamma(G)$ and $\omega(G) - \gamma(G)$ can be arbitrarily large for regular self-complementary graphs. We also prove the Hadwiger conjecture is true for self-complementary graphs. Furthermore, we prove that the Hadwiger number for regular self-complementary graph of order n is $\lfloor (n+1)/2 \rfloor$. For the class of self-complementary digraphs, we disprove a conjecture made in 1979 that every strongly connected self-complementary digraph has a Hamiltonian dipath. We also show that self-complementary graphs with arbitrarily large vertex connectivity do not satisfy this conjecture.

Keywords: self-complementary graphs, computational complexity, anti-morphism, complementing permutation, Hadwiger conjecture, Hadwiger number, self-complementary digraphs.

1   Introduction

Self-complementary graphs (sc-graphs in short) are one of the well studied graph classes. This is probably due to the algebraic insight into the adjacency of its vertices, found in the very first, and seminal paper [1] on this topic by Sachs in 1962 (also, later, independently by

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Ringel [2]). This technique is further elaborated by Gibbs [3] and Clapham [4]. In the past five decades, hundreds of papers have exploited this technique and proved many noteworthy results on sc-graphs. Graph coloring and maximum clique size are the most important and applied graph problems, and belong to the first identified $\mathcal{NP}$-complete problems in graphs. Moreover the independent set problem can be polynomially related to the maximum clique problem. In spite of the abundance of rich results on sc-graphs, complexity of finding these important graph parameters is not yet known. In the first half of this paper, we aim at classifying the computational complexity of finding such parameters when restricted to sc-graphs. In particular, we present surprisingly simple proofs of $\mathcal{NP}$-hardness of finding the chromatic number, clique number and independence number of sc-graphs. On the contrary we prove that for sc-graphs, checking whether the chromatic number is less than or equal to an integer k, for $k \geq 3$; and whether the clique number and the independence number are greater than or equal to an integer k, for $k \geq 3$, are in $\mathcal{P}$.

In this paper all graphs and digraphs, unless mentioned otherwise, are simple and finite. For definitions on graphs, we refer to Harary [5]; for complexity, we refer to Garey and Johnson [6]; and for sc-graphs, we refer to the extensive survey by Farrugia [7]. Let $G(V, E)$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$. It is a sc-graph if G is isomorphic to its complement $\overline{G}$. So the number of edges $m = n(n-1)/4$, and hence $n \equiv 0, 1 \pmod 4$. An anti-morphism σ is an isomorphism between a graph and its complement that exchanges edges and non-edges. An anti-morphism of a sc-graph can be expressed as a permutation σ on $V(G)$ i.e. $\sigma(G) = \overline{G}$. In this article we refer σ as a complementing permutation (c.p.) of G. For vertices $v, w \in G$, $v \sim w$ in G if and only if $\sigma(v) \sim \sigma(w)$ in $\overline{G}$. As usual the c.p. can be expressed as a product of disjoint cycles of the permutation. A sc-graph may have several c.p.s, and non-isomorphic sc-graphs may have the same c.p. The c.p. adds a few restrictions on the length of these cycles. For further details please refer [7].

It is easy to construct certain families of sc-graphs. For example, given any graph G, construct $G–\overline{G}–\overline{G}–G$, where $G–H$ have all vertices and edges of G and H, along with all edges between $V(G)$ and $V(H)$. It can be easily checked that this is a sc-graph. We call this operation as the $P_4 - construction$ of G. A slight variation of the $P_4 - construction$ of G is the $A - construction$ of G, where we add a new vertex to the $P_4 - construction$ of G and join it to all vertices of the two $\overline{G}$. This also results in a sc-graph. For other such constructions of sc-graphs, please refer [7, p. 13].

As we shall see later, *lexicographic products* play an important role in constructing bigger sc-graphs. Let G and H be graphs with vertex sets $\{a, b, ...\}$ and $\{x, y, ...\}$ respectively. The lexicographic product $G \circ H$ has vertex set $V(G) \times V(H)$, and $(a, x)(b, y) \in E(G \circ H)$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$. It turns out that lexicographic products of sc-graphs result in another sc-graph. Using this we construct a family of regular sc-graphs with increasing $\chi(G) - \omega(G)$, thus allowing it to be arbitrarily large.

The sc-graphs either have diameter 2 or 3. Finding the domination number $\gamma(G)$ of sc-graphs with diameter 3 is relatively easy. Paley graphs are regular sc-graphs with diameter 2. Using Paley graphs of a certain kind and some of its properties, we conclude that $\gamma(G)$ and the differences $\chi(G) - \gamma(G)$ & $\omega(G) - \gamma(G)$ can be arbitrarily high for regular sc-graphs.

Let $(v_1, v_2, ..., v_r)$ be a cycle of a c.p. with $p \geq 2$ of an sc-graph G. Without loss of generality assume that $v_1 v_2 \in E(G)$, then because of the c.p. $v_2 v_3 \in E(\overline{G})$, $v_3 v_4 \in E(G)$ and so on. Hence $v_{r-1} v_r$ should belong to $E(G)$ (else $v_1 v_r \in E(G)$, which contradicts $v_1 v_2 \in E(G)$). So r is even. This cycle $(v_1, v_2, ..., v_r)$ is a c.p. of a smaller induced sc-graph, which implies $r \equiv 0, 1 \pmod 4$. Since r is even, $r = 4k$, i.e. each of the cycle lengths are a multiple of 4. For a sc-graph on $4k$ vertices, we can divide $4k$ into cycles of lengths in multiples of 4. In fact different c.p. produce cycles of different lengths, but which are multiples of 4. However for a sc-graph of order $4k + 1$, we have a single fixed vertex and the lengths of the rest of cycles are multiples of 4. As a result, a sc-graph of order $4k + 1$ has at least one vertex with degree $2k$. Also different c.p.s result in different fixed vertices. Not only that, the c.p. provides a deeper restriction on the structure of the sc-graph. Since σ is a c.p., σ^2 is an automorphism. It is easy to see that σ^{odd} are c.p.s and σ^{even} are automorphisms. In any cycle of c.p. σ, the vertices alternate in degrees, and the sum of the consecutive degrees is $n - 1$. Now consider a cycle of length $4r$, it will have $2r$ pairs of vertices with complementing degrees. The sc-graph induced on such a cycle has odd vertices $(o_1, o_2, .., o_{2r})$ and even vertices $(e_1, e_2, .., e_{2r})$. Due to the automorphism σ^2, the neighbours of one vertex determines the neighbours of other vertices in the group. In the sc-graph there will be $2r^2$ edges — half of possible edges — between these groups.

It is obvious that, for any graph G, $\chi(G) \geq \omega(G)$, and this difference can be arbitrarily large (seen easily from the Mycielski's construction [8]). Although a graph with chromatic number k need not have a clique of size k, in [9], Hadwiger conjectured that it must be contractible to a clique of size k i.e. if $\chi(G) = k$, then $G > K_k$. For $k = 5$, Hadwiger conjecture implies the Four Color Theorem. Thus Hadwiger conjecture is considerably stronger that the Four Color Theorem. The Hadwiger number $\eta(G)$ of a graph G is the size k of the largest clique, known as the Hadwiger clique, obtained by contracting edges of G. This is equivalent to the largest k such that the vertices of G can be partitioned into k partitions $V_1, V_2, ..., V_k$, each inducing a connected subgraph, such that between any two sets of the partition there is an edge of G. Thus the Hadwiger conjecture can be stated as follows:

Conjecture 1.1. *(Hadwiger) For any graph* G, $\eta(G) \geq \chi(G)$.

Using the adjacency of vertices just described in sc-graphs, we prove that Hadwiger conjecture is true for sc-graphs. Furthermore, we find the Hadwiger number for regular sc-graphs, and also for chordal sc-graphs, of order n is $\lfloor (n + 1)/2 \rfloor$.

By a complete symmetric digraph $K_n^*(V, A)$, we mean that for every pair of vertices $u, v \in V(K_n^*)$, both directed arcs uv (read u to v) & $vu \in A(K_n^*)$. A sc-digraph D is isomorphic to its complement with respect to K_n^* i.e. $D \cong K_n^* - D$. A digraph is strongly connected if there are directed paths uv and vu for any two vertices u and v. In 1979, it was conjectured in [10] (see also [7, p. 58]) on the existence of Hamiltonian dipaths in strongly connected sc-digraphs. We disprove this conjecture, by explicitly constructing an infinite set of counterexamples with arbitrarily large vertex connectivity.

The Ramsey number $R(k, k)$ is the smallest n such that for any graph G of order n, either G or $\overline{G}$ contains a K_k i.e. G contains a K_k or $\overline{K_k}$. Of course any sc-graph of order at least $R(k, k)$ must contain a K_k. However the converse is not true. This can be easily verified on a P_4 − construction of $\overline{K_k}$. The following stronger conjecture was made by Chvátal et al. [11].

Conjecture 1.2. *Let $n^*(k)$ be the greatest n for which there exists at least one sc-graph of order n which does not contain a K_k. Then $n^*(k) = R(k, k) - 1$.*

For an extensive study on Ramsey numbers, please refer [12].

For a graph G and a set of vertices $S \subseteq V$, an induced subgraph $G[S]$ has vertex set S and edges whose both endpoints are in S.

Nordhus and Gaddum, in their well known paper [13], proved the following bounds on sum of the chromatic number of a graph and its complement: $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. So for a sc-graph $\sqrt{n} \leq \chi(G_{sc}) \leq (n + 1)/2$.

Below we list the problems whose computational complexity has been analysed or reviewed in this paper.

- MAXCLIQUE: Given a graph G, find its clique number $\omega(G)$.

- MAXIND: Given a graph G, find its independence number $\alpha(G)$.

- CHROMATIC: Given a graph G, find its chromatic number $\chi(G)$.

- CLIQUE: Given a graph G and an integer k, check whether G contains a set S of pairwise adjacent vertices with $|S| \geq k$.

- INDSET: Given a graph G and an integer k, check whether G contains a set S of pairwise non-adjacent vertices with $|S| \geq k$.

- k-CHROMATIC: Given a graph G and an integer k, check whether G is k-colorable.

- 3-CHROMATIC: Given a graph G and an integer k, check whether G is 3-colorable.

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- HAMPATH: Given a graph G, check whether G has a path that goes through every vertex of the graph.

- HAMCYCLE: Given a graph G, check whether G has a cycle that goes through every vertex of the graph.

- 2-FACTOR: Given a graph G, check whether G has collection of disjoint cycles containing all vertices of the graph.

- ISOMORPHISM: Given graphs G and H, check whether they are isomorphic.

- RECOGNITION: Given a graph G and a class of graphs, check whether G belongs to that class.

ISOMORPHISM is an important problem in algorithmic graph theory. For general graphs we do not know whether it belongs to $\mathcal{P}$ or $\mathcal{NP}$-complete. We say a problem is $\mathcal{GI}$-complete if it can be polynomially reduced to ISOMORPHISM.

Motivation: The study of the sc-graphs is motivated due to the following factors.

- Because of the c.p.s and automorphisms, we have a good understanding of the adjacency of vertices in sc-graphs, hence it is natural to expect certain problems on graph parameters will be easier to solve when restricted to sc-graphs. Identifying such problems and their characterisation, if possible, makes sc-graphs worth investigating.

- sc-graphs are effectively used to find lower bounds of Ramsey numbers $R(k, k)$ [14, 15, 16, 17, 18].

- sc-graphs serve as counter examples to several conjectures: C_5 being the smallest non-perfect graph, the line graph $L(K_{3,3} - e)$ being the smallest perfect graph with no even pair in it or its complement, and others [19].

Organization: Section 2 deals with the main theme of this paper i.e. computational complexity results on sc-graphs. We first review some of the standard complexity results on sc-graphs. Then we prove $\mathcal{NP}$-hardness of MAXCLIQUE, MAXIND and CHROMATIC for sc-graphs. Surprisingly, next we prove that CLIQUE, INDSET and k-CHROMATIC for sc-graphs are in $\mathcal{P}$. We end this section by showing that the the domination number $\gamma(G)$ as well as the differences $\chi(G) - \omega(G)$, $\chi(G) - \gamma(G)$ & $\omega(G) - \gamma(G)$ can be arbitrarily large for regular sc-graphs. In Section 3, first we prove the Hadwiger conjecture for sc-graphs, following which we find the Hadwiger number for regular sc-graphs and chordal sc-graphs. In Section 4, we disprove the conjecture on the existence of Hamiltonian dipaths in strongly connected

sc-digraph, by explicitly constructing an infinite set of counterexamples with arbitrarily large vertex connectivity. Section 5 contains a few open problems we touch upon in this paper, ending with some concluding remarks.

2   Complexity Results

Apart from the structural insights, the use of c.p.s has played a pivotal role in solving many of the problems regarding sc-graphs. Clapham has used this technique to solve HamPath in sc-graphs [4], as well as for certain infinite sc-graphs [20]. Rao has repeatedly used this technique to determine the lengths of cycles in sc-graphs [21], a good characterization (including recognition) of 2-Factors in sc-graphs [22] and good characterisation (including recognition) of HamCycle in sc-graphs [23]. These were used to find polynomial algorithms [23, 24] for 2-Factors, HamPath and HamCycle in sc-graphs, whenever they exist. Now we look into some of the difficult problems concerning sc-graphs.

As discussed previously, Isomorphism is one of the most important problems in graph theory, which deals with distinguishing members of a class from one another. Recognition is also another fundamental problem for any class of graphs, which deals with distinguishing members of a class from non-members. There is an important connection between Isomorphism for graphs and Recognition for sc-graphs. Had the Isomorphism been polynomial, we would have solved Recognition for sc-graphs, by just checking whether the graph and its complement are isomorphic or not. In [25] Colbourn and Colbourn proved that if there is a polynomial algorithm for Recognition for sc-graphs then Isomorphism for graphs will be polynomial, hence proving Recognition for sc-graphs is polynomially equivalent to Isomorphism for graphs. It is natural to expect that the Isomorphism would be easier when restricted to sc-graphs, due to the rich understanding of the adjacency of their vertices. However in [26], Colbourn and Colbourn prove that Isomorphism for sc-graphs is also polynomially equivalent to Isomorphism for graphs. So both the Recognition and Isomorphism for sc-graphs are $\mathcal{GI}$-complete. Apart from the sc-graph perspective, these problems are a window to the famous $\mathcal{P}$ v/s $\mathcal{NP}$ problem, due to the following theorem by Farrugia [7, p. 99].

Theorem 2.1. $\mathcal{P} = \mathcal{NP}$ *iff* Recognition *or* Isomorphism *for sc-graphs is* $\mathcal{NP}$-*complete.*

2.1   New Complexity Results for sc-graphs

Now we shall prove $\mathcal{NP}$-hardness of MaxClique (hence MaxInd) and Chromatic for sc-graphs. We first prove that these problems are $\mathcal{NP}$-hard on $\mathcal{G}_{sc}$, the class of sc-graphs $G_{sc} = P_4 - $ construction of any graph G. Using this we prove the following theorem.

Theorem 2.2. MAXCLIQUE *is* $\mathcal{NP}$-*hard for sc-graphs.*

In order to prove Theorem 2.2 we need the following lemma.

Lemma 2.3. MAXCLIQUE *is* $\mathcal{NP}$-*hard for* $\mathcal{G}_{sc}$.

Proof. Clearly $\omega(G_{sc}) = \max\{\alpha(G) + \omega(G), 2\alpha(G)\}$. If we consider G to be Δ-free, excluding a few finite graphs (with $\omega(G) = \alpha(G) = 2$), $\omega(G) < \alpha(G)$ and hence $\omega(G_{sc}) = 2\alpha(G)$. MAXIND for Δ-free graphs is known to be $\mathcal{NP}$-hard [27]. Hence MAXCLIQUE is $\mathcal{NP}$-hard for $\mathcal{G}_{sc}$. $\square$

Proof. (of Theorem 2.2) By restriction, Lemma 2.3 proves the theorem. $\square$

Since clique number and independence number of sc-graphs are the same, we have the following theorem.

Theorem 2.4. MAXIND *is* $\mathcal{NP}$-*hard for sc-graphs.*

The following theorem addresses the complexity of finding chromatic number of a sc-graph.

Theorem 2.5. CHROMATIC *is* $\mathcal{NP}$-*hard for sc-graphs.*

Proof. Clearly $\chi(G_{sc}) = \max\{\chi(G) + \chi(\overline{G}), 2\chi(\overline{G})\}$. We consider G to be a planar graph with order greater than 25. Using the Nordhus-Gaddum relation $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$ and the five color theorem for planar graphs, we have $\chi(G) < \chi(\overline{G})$, hence $\chi(G_{sc}) = 2\chi(\overline{G})$. Since CHROMATIC for co-planar graphs is $\mathcal{NP}$-hard [28] (Clique cover is $\mathcal{NP}$-complete for planar graphs), so is CHROMATIC for $\overline{G}$ (since we exclude a finite list of graphs). So CHROMATIC is $\mathcal{NP}$-hard for $\mathcal{G}_{sc}$. Hence by restriction, CHROMATIC is $\mathcal{NP}$-hard for sc-graphs. $\square$

Remark 2.6. *As a result of the strong perfect graph theorem [29], we have the following characterization of perfect sc-graphs. A sc-graph is perfect if it has no induced odd cycles of length greater that equal to 5.*

Although CHROMATIC for sc-graph is $\mathcal{NP}$-complete, the following theorem states that k-CHROMATIC is in $\mathcal{P}$.

Theorem 2.7. k-CHROMATIC *is in* $\mathcal{P}$ *for sc-graphs.*

Proof. Due to the Nordhus-Gaddum relation, for a sc-graph G, we have the following bounds: $\sqrt{n} \leq \chi(G) \leq (n+1)/2$. This implies that there are only finite number of sc-graphs that are r-partite (also see Faruggia [7, p. 33]). As a result the number of sc-graphs with

chromatic number r is also finite (sc-graphs of order less than r^2). This means for $k \leq r$, we have a finite list of sc-graphs, whose individual chromatic numbers can be found out in polynomial time. Hence k-CHROMATIC is in $\mathcal{P}$ for sc-graphs. $\square$

Remark 2.8. *It is interesting to note that for a given k and a sc-graph G we can answer, in polynomial time, whether $\chi(G) \leq k$ or not, however finding that $\chi(G) = k$ turns out to be $\mathcal{NP}$-hard.*

In particular we have the following corollary.

Corollary 2.9. 3-CHROMATIC *is in $\mathcal{P}$ for sc-graphs.*

We present similar results for clique number and independence number of sc-graphs.

Theorem 2.10. CLIQUE *is in $\mathcal{P}$ for sc-graphs.*

Proof. Since $\omega(G) \leq \chi(G)$, we have for a sc-graph $\omega(G) \leq (n+1)/2$. For a sc-graph of order n the clique number $\omega(G) \geq p$, where p is maximal such that $R(p, p) \leq n$. We know the existence of such a p (since $R(p, p) \leq \binom{2p-2}{p-1}$ [12]), for every n. So we have $p \leq \omega(G) \leq (n+1)/2$. So there are finite number of sc-graph that have clique number r. Hence for $k \leq r$, there is a finite list of sc-graphs, whose individual clique numbers can be found out in polynomial time. Hence CLIQUE is in $\mathcal{P}$ for sc-graphs. $\square$

Since clique problem and independent set problem is same for sc-graphs, we have the following theorem.

Theorem 2.11. INDSET *is in $\mathcal{P}$ for sc-graphs.*

2.2 Other Problems

2.2.1 $\chi(G) - \omega(G)$ is arbitrarily large for regular sc-graphs

We are interested in lexicographic products because of the following lemma.

Lemma 2.12. *The lexicographic product of two sc-graphs is a sc-graph.*

Proof. Let G_{sc} and H_{sc} be two sc-graphs with vertex set $\{u_i\}$ and $\{v_i\}$, and with c.p. σ_g and σ_h respectively. We prove that the c.p. of the lexicographic product $G_{sc} \circ H_{sc}$, $\sigma_{gh}(u_i, v_j) = (\sigma_g(u_i), \sigma_h(v_j))$. First we show that if edge $(u_{i1}, v_{j1})(u_{i2}, v_{j2}) \in E(G_{sc} \circ H_{sc})$ then $(\sigma_g(u_{i1}), \sigma_h(v_{j1}))(\sigma_g(u_{i2}), \sigma_h(v_{j2})) \notin E(\overline{G_{sc} \circ H_{sc}})$. If $(u_{i1}, v_{j1})(u_{i2}, v_{j2}) \in E(G_{sc} \circ H_{sc})$ then either $u_{i1} = u_{i2}$ and $v_{j1}v_{j2} \in E(H)$, or $u_{i1}u_{i2} \in E(G)$. This implies either $\sigma_g(u_{i1}) = \sigma_g(u_{i2})$

and $\sigma_h(v_{j1})\sigma_h(v_{j2}) \notin E(H)$, or $\sigma_g(u_{i1})\sigma_g(u_{i2}) \notin E(G)$ (because σ_g and σ_h are c.p.s). So $(\sigma_g(u_{i1}), \sigma_h(v_{j1}))(\sigma_g(u_{i2}), \sigma_h(v_{j2})) \notin E(\overline{G_{sc} \circ H_{sc}})$. Similarly we see that if $(u_{i1}, v_{j1})(u_{i2}, v_{j2}) \notin E(G_{sc} \circ H_{sc})$ then we have the following: $(\sigma_g(u_{i1}), \sigma_h(v_{j1}))(\sigma_g(u_{i2}), \sigma_h(v_{j2})) \in E(\overline{G_{sc} \circ H_{sc}})$. Hence $G_{sc} \circ H_{sc}$ is a sc-graph. $\qquad\square$

In particular we have the following corollary.

Corollary 2.13. *If G_{sc} is a sc-graph, so is $G_{sc} \circ G_{sc}$.*

Now we look into the clique number and chromatic number of lexicographic products. It is well known that $\omega(G \circ H) = \omega(G).\omega(H)$. We have a theorem due to Geller and Stahl [30] on the chromatic number of lexicographic products, which states that if $\chi(H) = n$, then $\chi(G \circ H) = \chi(G \circ K_n)$. Using this we prove the next theorem by an explicit construction.

Theorem 2.14. *The difference between chromatic number and clique number of a regular sc-graph can be arbitrarily large.*

Proof. We look into chromatic number of $\mathcal{P} \circ \mathcal{P}$ where $\mathcal{P}$ is a pentagon. Let the topmost vertex of the pentagon be labelled 1 and increase the labels in a clockwise manner. For $\mathcal{P}^2 = \mathcal{P} \circ \mathcal{P}$ we number vertices of topmost pentagon as $(11, ..., 15)$ and so on till $(51, ..., 55)$. We continue this labelling to higher powers of $\mathcal{P}$. For $\mathcal{P}^n$ let P_1 represent the topmost $\mathcal{P}^{n-1}$ and increase in a clockwise manner till P_5. It is easy to check that $\mathcal{P}^n$ is a regular sc-graph.

Consider $\mathcal{P}^2$, P_1 to P_5 are pentagons with chromatic number $\chi(\mathcal{P}) = 3$. For $P_i P_{i+1}$ we need 6 colors. Hence we give a proper coloring to P_1 with colors $(1, 2, 3)$ and P_2 with colors $(4, 5, 6)$. We can give colors $(7, 8, 9)$ to P_5 and P_3, and $(4, 5, 6)$ to P_4. But this would not be optimal. For an optimal coloring we give colors $(4, 7, 8)$ to P_5, and for P_3 and P_4 we have colors $(1, 2, 3, 5, 6, 7, 8)$. However if attempt to use 7 colors and use $(4, 5, 7)$ for P_5 wo would have $(1, 2, 3, 6, 7)$ for P_3 and P_4 and hence a proper coloring would not be possible.

Similarly for $\mathcal{P}^n$, P_1 to P_5 are $\mathcal{P}^{n-1}$s with chromatic number $\chi(\mathcal{P}^{n-1})$. For $P_i P_{i+1}$ we need $2\chi(\mathcal{P}^{n-1})$ colors. Hence we give a proper coloring to P_1 with colors $(1, 2, 3)$ and P_2 with colors $(4, 5, 6)$. For an optimal coloring we give $\lceil\chi(\mathcal{P}^{n-1})/2\rceil$ new colors to P_5 and for the rest of P_5 use the colors of P_2, we will have $\chi(\mathcal{P}^{n-1}) + 2\lceil\chi(\mathcal{P}^{n-1})/2\rceil = 2\chi(\mathcal{P}^{n-1}) + 1$ colors for P_3 and P_4. So $\chi_n = 2\chi(\mathcal{P}^{n-1}) + \lceil\chi(\mathcal{P}^{n-1})/2\rceil = \lceil 5\chi(\mathcal{P}^{n-1})/2\rceil$. If we had used $\lceil\chi(\mathcal{P}^{n-1})/2\rceil - 1$ new colors in P_5 we will be left with $\chi(\mathcal{P}^{n-1}) + 2\lceil\chi(\mathcal{P}^{n-1})/2\rceil - 2 = 2\chi(\mathcal{P}^{n-1}) - 1$ colors for P_3 and P_4, which is not possible. So we cannot get a proper coloring on $2\chi(\mathcal{P}^{n-1}) + \lceil\chi(\mathcal{P}^{n-1})/2\rceil - 1$ colors. Hence $\chi(\mathcal{P}^n) = \lceil 5\chi(\mathcal{P}^{n-1})/2\rceil$.

We know that $\omega(\mathcal{P}) = 2$ and $\omega(\mathcal{P}^n) = 2\omega(\mathcal{P}^{n-1}) = 2^n$. Clearly $\chi(\mathcal{P}) > \omega(\mathcal{P})$, and with $\chi(\mathcal{P}^n) = \lceil 5\chi(\mathcal{P}^{n-1})/2\rceil$ and $\omega(\mathcal{P}^n) = 2\omega(\mathcal{P}^{n-1})$, we conclude that $\omega(\mathcal{P}^n) - \chi(\mathcal{P}^n) \to \infty$ as $n \to \infty$. So the difference between $\chi(G)$ and $\omega(G)$ of regular sc-graphs can be arbitrarily large. $\qquad\square$

2.2.2 Domination Number

Now we look into another important graph parameter: the domination number of sc-graphs. We begin with the following lemma (for proof see [7, p. 4]) and the corresponding corollary.

Lemma 2.15. *For a sc-graph G_{sc}, $\text{diam}(G_{sc}) = 3$ iff it has a dominating edge.*

This naturally leads to the following corollary.

Corollary 2.16. *sc-graphs with diameter 3 have domination number 2.*

So we need to look into sc-graphs with diameter 2 i.e. self centered sc-graphs. Using the examples of the self centered sc-graphs constructed by the traditional methods, we mostly find that two vertices are enough to dominate such sc-graphs. This is probably due to the abundance of connections from these vertices. Because every vertex in such sc-graphs have eccentricity 2, these examples suggest we can cover such sc-graphs just by two stars. However we also found sc-graphs with domination number 3, for example $\gamma(\mathcal{P}^n)$ for $n \geq 2$. The following lemma illustrates a class of sc-graphs which can have large domination numbers.

Lemma 2.17. *Paley graphs can have arbitrarily large domination numbers.*

Proof. As shown by Lee [31], the domination number of Paley graph G of order n satisfies $(1/2 - \epsilon)\log n \leq \gamma(G) \leq \log n + 1$. So domination number of Paley graphs can be arbitrarily large. $\qquad\square$

So we have the following corollary summarizing results on domination numbers of sc-graphs.

Corollary 2.18. *Domination number of regular sc-graphs can be arbitrarily large.*

2.2.3 $\chi(G) - \gamma(G)$ & $\omega(G) - \gamma(G)$ is arbitrarily large for regular sc-graphs

The maximum independent set is a dominating set, so $\gamma(G) \leq \alpha(G)$. Hence in case of sc-graphs we have $\gamma(G) \leq \omega(G) \leq \chi(G)$. Therefore we look into the differences $\chi(G) - \gamma(G)$ & $\omega(G) - \gamma(G)$. Due to Broere, Doman and Ridley [32], it is known that the chromatic number and clique numer of Paley graphs with order p, an even power of a prime, are both $\sqrt{p}$, whereas due to Lee [31] we have the upperbound of the domination number of Paley graphs with order p is $\log p + 1$. Since Paley graphs are regular sc-graphs, we conclude $\chi(G) - \gamma(G)$ and $\omega(G) - \gamma(G)$ can also be arbitrarily large for regular sc-graphs.

3 Further Results on sc-graphs

In this section, we prove the Hadwiger Conjecture for sc-graphs. A theorem without proof was mentioned by Rao [10, Thm. 4.1] (also see [7, p. 38]) and a formal proof was given by Girse and Gillman [33]. We present a simple constructive proof.

Theorem 3.1. *For a sc-graph of order* n

$$\chi(G) \leq \left\lfloor \frac{n+1}{2} \right\rfloor \leq \eta(G).$$

In particular, the Hadwiger conjecture is true for sc-graphs.

Proof. Without loss of generality, we assume that in each cycle of the c.p. $(v_1, v_2, ..., v_{4r})$ of sc-graph G, $v_{2i-1}v_{2i} \in E(G)$, where $i \leq 2r$ is a non-negative integer. And let v_0 be the fixed point of any sc-graph of odd order. We consider both the cases where order of the sc-graph is $4k$ or $4k+1$.

If order of the sc-graph is $4k$, from the Nordhus-Gaddum relation we have $\chi(G) \leq (4k+1)/2$, i.e. $\chi(G) \leq 2k$. Assuming the edges $v_{2i-1}v_{2i}$ to be the subgraphs, there are $2k$ subgraphs in total. It is easy to see that between any two edges $v_{2i-1}v_{2i}$ and $v_{2j-1}v_{2j}$, either of $v_{2i-1}v_{2j-1}$ or $v_{2i}v_{2j} \in E(G)$. Hence Hadwiger number $\eta(G) \geq 2k$.

If order of the sc-graph is $4k+1$, from the Nordhus-Gaddum relation we have $\chi(G) \leq (4k+1+1)/2$, i.e. $\chi(G) \leq 2k+1$. Assuming the edges $v_{2i-1}v_{2i}$ and vertex v_0 to be the subgraphs, there are $2k+1$ subgraphs in total. It is easy to see that between any two edges $v_{2i-1}v_{2i}$ and $v_{2j-1}v_{2j}$, either of $v_{2i}v_{2j}$ or $v_{2i-1}v_{2j-1} \in E(G)$ Also between v_0 and $v_{2i-1}v_{2i}$, either v_0v_{2i-1} or $v_0v_{2i} \in E(G)$. Hence Hadwiger number $\eta(G) \geq 2k+1$. So the Hadwiger conjecture is true for sc-graphs. □

In the following lemma, we prove that the lower bound of the Hadwiger number $\eta(G)$ is attained by regular sc-graphs.

Lemma 3.2. *Hadwiger number of regular sc-graphs of order* n *is* $\lfloor (n+1)/2 \rfloor$.

Proof. It is evident that regular sc-graphs have odd order, say $4k+1$, making the degree of each vertex $2k$. Using Theorem 3.1 we have $\eta(G) \geq 2k+1$. So in any $\eta(G)$ partition of the vertex set $V(G)$, there is a set V_i which is singleton; say $V_i = \{v_0\}$. Since degree of v_0 is $2k$ and it is joined to all sets in the partition, we have $\eta(G) \leq 2k+1$ (v_0 and its $2k$ neighbours). Hence $\eta(G) = 2k+1 = \lfloor (n+1)/2 \rfloor$. □

Remark 3.3. *Using results on chromatic number, clique number and domination number of a class of Paley graphs given in Subsections 2.2.2 & 2.2.3, and observing that Paley graphs are*

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regular sc-graphs, we conclude that the differences $\eta(G) - \chi(G)$, $\eta(G) - \omega(G)$ *and* $\eta(G) - \gamma(G)$ *can be arbitrarily large for regular sc-graphs.*

We further observe that this value is attained by a few other classes of sc-graphs, with subgraphs as mentioned in the proof of Theorem 3.1. We shall now prove that chordal sc-graphs also have this Hadwiger number.

Lemma 3.4. *Hadwiger number of chordal sc-graphs of order* n *is* $\lfloor (n+1)/2 \rfloor$.

Proof. Stiebitz [34] proved that for a chordal graph, we have $\eta(G) = \omega(G)$. Further applying Theorem 3.1, we conclude that Hadwiger number of chordal sc-graphs of order n is $\lfloor (n+1)/2 \rfloor$. $\qquad \square$

Due to Kostochka [35], we have the following Nordhaus–Gaddum type result on Hadwiger numbers of graphs

$$\eta(G) + \eta(\overline{G}) \leq \left\lfloor \frac{6n}{5} \right\rfloor.$$

Hence for a sc-graph $\eta(G) \leq \frac{1}{2} \left\lfloor \frac{6n}{5} \right\rfloor$.

Below we give a construction, inspired by one given in [34], to produce sc-graphs with Hadwiger numbers varying from $\lfloor (n+1)/2 \rfloor$ to $\lfloor 6n/5 \rfloor /2$.

Construction: Let X_1, X_2, X_3, X_4 and X_5 be disjoint sets of vertices satisfying $\mid X_i \mid = q$, for $i = 2$ to 5, and $\mid X_1 \mid = r$, for integers q, $r > 0$ and $r \equiv 0, 1 \pmod 4$. Let X_1 induce a sc-graph of order r, X_3 and X_4 induce complete graphs, and X_2 and X_5 induce empty graphs. Join X_i to X_{i+1}, for $i = 1$ to 4, and X_5 to X_1 by all possible edges. The resulting graph G on $4q + r$ vertices is a sc-graph. It also can be easily checked that its Hadwiger number

$$\eta(G) = \begin{cases} 2q + r & r \leq q \\ 3q & r > q, \ \eta(G[X_1]) \leq q \\ 2q + \eta(G[X_1]) & r > q, \ \eta(G[X_1]) > q. \end{cases}$$

So for given order $n \equiv 0, 1 \pmod 4$, we can construct sc-graphs whose Hadwiger numbers attain the bounds. Whether all intermediate values are attained is a problem for future research.

Remark 3.5. *We can easily construct a sc-graph with given Hadwiger number* $\eta(G)$. *If* $\eta(G)$ *is even, then we construct a regular sc-graph of order* $2\eta(G) + 1$. *If* $\eta(G)$ *is odd, then we do the above construction with* $q = \eta(G)$ *and* $r = 1$, *resulting in a sc-graph of order* $4\eta(G) + 1$.

4   Results on sc-digraphs

The sc-digraphs are one of the lesser studied parts of the theory of sc-graphs. Here we disprove the following conjecture made by the first author in [10] (see also [7, p. 58]).

Conjecture 4.1. *Every strongly connected sc-digraph has a Hamiltonian dipath.*

Proof. We disprove Conjecture 4.1 by constructing an infinite set of counterexamples. Construct a graph G such that its vertex set $V(G)$ is partitioned into A_1, A_2, A_3 and A_4, such that $|A_1| = |A_2| = n$ and $|A_3| = |A_4| = m$. Furthermore $G[A_1 \cup A_3]$ is a complete symmetric digraph where as $G[A_2 \cup A_4]$ is an empty graph on $n+m$ nodes. We also have the following additional edges $A_i - A_{i+1}$ under addition modulo 4, where G—H means every vertex in H is adjacent to every vertex of G. It is easy to see that G is a strongly connected sc-digraph. Let $m > 2n$ and $n \geq 2$. Assume that there is a Hamiltonian dipath in G. For every vertex $u \in A_4$, the next vertex on P will be in A_1, unless u is the last vertex of P. Since $m > 2n$, this is not possible. So G has no Hamiltonian dipath. By taking various values of $n \geq 2$ we can construct an infinite class of counter-examples. $\square$

Remark 4.2. *It is easy to note that the vertex connectivity of G constructed in the above proof is n. Hence large vertex connectivity does not in general imply the existence of Hamiltonian dipath in a sc-digraph.*

5   Open Problems and Conclusion

Below we list some of the open problems encountered in this paper.

1. Characterise sc-digraphs with Hamiltonian dipaths.

2. Characterise sc-digraphs with Hamiltonian dicycles.

3. Characterise sc-digraphs with directed 1-factor.

4. Determine complexity of DOMINATION for sc-graphs.

5. Determine the possible Hadwiger numbers of sc-graphs of order n.

Please note that the problems 1 to 3 are settled for sc-graphs. It is evident from the above list that, although a lot of work has been done in sc-graphs, many of the corresponding problems are not yet analysed for sc-digraphs.

From our results, we can design polynomially bounded algorithms for solving some of the well known decision problems on graph parameters, yet these algorithms are not practical.

On the $\mathcal{NP}$-hardness of finding the chromatic number, clique number and independence number, it is interesting to note that finding these important graph parameters is difficult even for sc-graphs, which is a very small fraction of graphs and for which we know the adjacency of its vertices very well.

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