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CHARACTERIZATION OF DISTRIBUTIONS BASED ON FUNCTIONS OF CONDITIONALLY INDEPENDENT RANDOM VARIABLES

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Abstract: Characterization problems or identifiability issues based on functions of conditionally independent random variables are studied.

1 Introduction

Properties of conditionally independent random variables were studied in Prakasa Rao (2009). Conditional versions of generalized Borel-Cantelli lemma, generalized Kolmogorov inequality, Hajek-Renyi inequality, strong law of large numbers and central limit theorem were discussed in Prakasa Rao (2009). Earlier discussions on the topic of conditional independence can be found in Chow and Teicher (1978) and Majerak et al. (2005). Roussas (2008) studied additional results for conditionally independent random variables. Bairamov (2011) investigated the copula representations for conditionally independent random variables and studied the distributional properties of order statstics of these random variables. Dawid (1979, 1980) observed that many important concepts in statistics can be considered as expressions of conditional independence. Shaked and Spizzichino (1998) considered n nonnegative random variables $T_i, i = 1, \ldots, n$ which are interpreted as the lifetimes of n units and assuming that $T_1, \ldots, T_n$, are conditionally independent, given some random variable Θ, determined the conditions under which $T_i, i = 1, \ldots, n$ are positively dependent. It is known that conditional independence of a set of random variables does not imply independence and independence does not imply conditional independence. This can be seen from the examples given in Prakasa Rao (2009).

We now discuss some characterization or identifiability problems for conditionally independent random variables. Analogous results for independent random variables were studied in Prakasa Rao (1992) following the works of Kotlarski and others. Through out this paper, we assume that the conditional distributions specified exist as regular conditional distributions. For brevity, we write "for all z" for the statement "for all z in the support of the distribution function of the random variable Z."

2 Identification of component distributions from joint distribution of sums

Suppose X_1, X_2 and X_3 are conditionally independent random variables given a random variable Z. Let $\phi_i(t; z)$ denote the conditional characteristic function of the random variable X_i given the event $Z = z$. Let $Y_1 = X_1 - X_3$ and $Y_2 = X_2 - X_3$.

Theorem 2.1: If the conditional characteristic function of the bivariate random vector (Y_1, Y_2) given $Z = z$ does not vanish, then the joint distribution of (Y_1, Y_2) given $Z = z$ determines the distributions of X_1, X_2 and X_3 given $Z = z$ up to a change in location depending on z.

Proof : Let $\phi(t_1, t_2; z)$ denote the conditional characteristic function of (Y_1, Y_2) given $Z = z$. Let $\phi_k(t; z)$ denote the conditional characteristic function of X_k given $Z = z$ for $k = 1, 2, 3$. Then, for any t_1, t_2 real,

$$\begin{aligned} \phi(t_1, t_2; z) &= E[\exp(it_1Y_1 + it_2Y_2)|Z = z] \\ &= E[\exp(it_1(X_1 - X_3) + it_2(X_2 - X_3)|Z = z] \\ &= E[\exp(it_1X_1 + it_2X_2 - i(t_1 + t_2)X_3)|Z = z] \\ &= \phi_1(t_1; z)\, \phi_2(t_2; z)\, \phi_3(-t_1 - t_2; z) \end{aligned} \tag{2. 1}$$

by the conditional independence of the random variables $X_i, 1 \leq i \leq 3$ given $Z = z$. Since $\phi(t_1, t_2; z) \neq 0$ for all t_1 and t_2 by hypothesis, it follows that $\phi_k(t; z) \neq 0$ for all t. Let W_1, W_2 and W_3 be another set of three conditionally independent random variables given Z with the conditional characteristic functions $\psi_k(t; z)$ given $Z = z$. Let $V_1 = W_1 - W_3$ and $V_2 = W_2 - W_3$ and $\psi(t_1, t_2; z)$ be the conditional characteristic function of the random vector (V_1, V_2) given $Z = z$. Suppose the conditional distributions of the random vectors (Y_1, Y_2) and (V_1, V_2) are the same given $Z = z$. Then, it follows that

$$\phi(t_1, t_2; z) = \psi(t_1, t_2; z), -\infty < t_1, t_2 < \infty \tag{2. 2}$$

for all z. Hence

$$\phi_1(t_1; z)\, \phi_2(t_2; z)\, \phi_3(-t_1 - t_2; z) = \psi_1(t_1; z)\, \psi_2(t_2; z)\, \psi_3(-t_1 - t_2; z) \tag{2. 3}$$

for all z. Furthermore $\phi_i(t; z) \neq 0, i = 1, 2, 3$ and $\psi_i(t; z) \neq 0, i = 1, 2, 3$ for all t real by hypothesis and for all z since $\phi(t_1, t_2; z) = \psi(t_1, t_2; z) \neq 0$ for all t_1, t_2 real and for all z. Let

(2. 4) $$\gamma_i(t; z) = \psi_i(t; z)/\phi_i(t; z), i = 1, 2, 3.$$

Note that the functions $\gamma_i(.; z), i = 1, 2, 3$ are continuous complex-valued functions with $\gamma_i(0; z) = 1, i = 1, 2, 3$ satisfying the equation

(2. 5) $$\gamma_1(t_1; z)\ \gamma_2(t_2; z)\gamma_3(-t_1 - t_2; z) = 1, -\infty < t_1, t_2 < \infty$$

for all z. Let $t_1 = t$ and $t_2 = 0$ in (2.5). Then we have

(2. 6) $$\gamma_1(t; z)\ \gamma_3(-t; z) = 1, -\infty < t < \infty$$

for all z. Let $t_2 = t$ and $t_1 = 0$ in (2.5). Then we have

(2. 7) $$\gamma_2(t; z)\ \gamma_3(-t; z) = 1, -\infty < t < \infty$$

for all z. Substituting for $\gamma_1(t; z)$, and $\gamma_2(t; z)$ in terms of $\gamma_3(t; z)$ in (2.5), it follows that

(2. 8) $$\gamma_3(t_1 + t_2; z) = \gamma_3(t_1; z)\ \gamma_3(t_2; z), -\infty < t_1, t_2 < \infty$$

with $\gamma_3(0; z) = 1$ for all z. It is known that the only measurable solution of this Cauchy functional equation is

(2. 9) $$\gamma_3(t; z) = e^{c(z)\ t}, -\infty < t < \infty$$

where $c(z)$ is a complex-valued function depending only on z. Observing that $\gamma_i(-t; z)$ is the complex conjugate of $\gamma_i(t; z)$ for all z from the properties of the characteristic functions, it is easy to see that

(2. 10) $$\gamma_1(t; z) = \gamma_2(t; z) = \gamma_3(t; z) = e^{c(z)t}, -\infty < t < \infty.$$

This equation in turn implies that

(2. 11) $$\psi_j(t; z) = \phi_j(t; z)\ e^{c(z)t}, -\infty < t < \infty, j = 1, 2, 3$$

for all z. Since $\psi_j(t; z)$ is the complex conjugate of $\psi_j(-t; z)$ from the properties of characteristic functions, it follows that $c(z) = i\ \beta(z)$ where $\beta(z)$ is a real-valued function. Therefore

(2. 12) $$\psi_j(t; z) = \phi_j(t; z)\ e^{i\beta(z)t}, -\infty < t < \infty, j = 1, 2, 3$$

for all z where $\beta(z)$ is a real-valued function depending only on z. From the properties of characteristic functions, it follows that the conditional distributions of the random variables W_j and $X_j + \beta(z)$ are the same for $j = 1, 2, 3$ given the event $Z = z$.

The result proved above gives sufficient conditions under which the conditional joint distributions of three conditionally independent random variables determine the conditional distributions of the individual summands. We now give a method which explicitly determine the distributions of the individual summands under some additional conditions.

3 Explicit determination of component distributions from joint distribution of sums

Suppose X_0, X_1, X_2 are conditionally independent random variables given a random variable Z with conditional characteristic functions $\phi_i(t;z), i = 0,1,2$ respectively given $Z = z$. Suppose that the characteristic functions $\phi_i(t;z)$ are different from zero for all $t \in R$. Let $Y_1 = X_0 + X_1$ and $Y_2 = X_0 + X_2$. Let $\psi(t_1, t_2; z)$ be the conditional characteristic function of (Y_1, Y_2) given $Z = z$. Suppose this function is *known*. It is obvious that

$$\psi(t_1, t_2; z) = \phi_0(t_1 + t_2; z)\ \phi_1(t_1; z)\ \phi_2(t_2; z), -\infty < t_1, t_2 < \infty \tag{3. 1}$$

for all z from the conditional independence of the random variables X_0, X_1, X_2 given $Z = z$. Note that $\psi(t_1, t_2; z) \neq 0$ for all $-\infty < t_1, t_2 < \infty$. Let $t_2 = 0$ in (3.1). Then we get that

$$\phi_0(t_1; z)\ \phi_1(t_1; z) = \psi(t_1, 0; z), -\infty < t_1 < \infty \tag{3. 2}$$

for all z from the properties of characteristic functions. Let $t_1 = 0$ in (3.1).Then we have

$$\phi_0(t_2; z)\ \phi_2(t_2; z) = \psi(0, t_2; z), -\infty < t_2 < \infty \tag{3. 3}$$

for all z. Relations (3.1) to (3.3) show that

$$\begin{aligned} &\phi_0(t_1 + t_2; z)\ \phi_1(t_1; z)\ \phi_2(t_2; z)\ \psi(t_1, 0; z)\ \psi(0, t_2; z) \\ = &\ \psi(t_1, t_2; z)\ \phi_0(t_1; z)\ \phi_1(t_1; z)\ \phi_0(t_2; z)\phi_2(t_2) \end{aligned} \tag{3. 4}$$

and hence

$$\phi_0(t_1 + t_2; z) = \frac{\psi(t_1, t_2; z)}{\psi(t_1, 0; z)\psi(0, t_2; z)}\phi_0(t_1; z)\phi_0(t_2; z) \tag{3. 5}$$

for $-\infty < t_1, t_2 < \infty$ and for all z. Let $\psi_i(t; z) = \log \phi_i(t; z)$ be the continuous branch of the logarithm of $\phi_i(t; z)$ with $\psi_i(0; z) = 0$. Then it follows that

$$\psi_0(t_1' + t_2; z) = \log \frac{\psi(t_1', t_2; z)}{\psi(t_1', 0; z)\psi(0, t_2; z)} + \psi_0(t_1'; z) + \psi_0(t_2; z) \tag{3. 6}$$

for $-\infty < t_1', t_2 < \infty$ and for all z. Assume that integration on both sides of the equation (3.6) with respect to t_1' over the interval $[0, t_1]$ is valid. Then it follows that

$$\text{(3. 7)} \qquad \int_0^{t_1} \psi_0(t_1' + t_2; z)dt_1' = \int_0^{t_1} \log \frac{\psi(t_1', t_2; z)}{\psi(t_1', 0)\psi(0, t_2; z)} dt_1' + \int_0^{t_1} \psi_0(t_1'; z)dt_1' + \int_0^{t_1} \psi_0(t_2; z)dt_1'$$

for $-\infty < t_1 < \infty$ and for all z. Let $t = t_1' + t_2$ in the integral on the leftside of the equation (3.7). Then it follows that

$$\text{(3. 8)} \qquad \int_{t_2}^{t_1+t_2} \psi_0(t; z)dt = \int_0^{t_1} \log \frac{\psi(t_1', t_2; z)}{\psi(t_1', 0)\psi(0, t_2; z)} dt_1' + \int_0^{t_1} \psi_0(t; z)dt + t_1\psi_0(t_2; z).$$

Rewriting the equation (3.6) in the form

$$\text{(3. 9)} \qquad \psi_0(t_1 + t_2'; z) = \log \frac{\psi(t_1, t_2'; z)}{\psi(t_1, 0; z)\psi(0, t_2'; z)} + \psi_0(t_1; z) + \psi_0(t_2'; z)$$

and integrating on both sides of this equation with respect to t_2' over the interval $[0, t_2]$, it follows that

$$\text{(3. 10)} \qquad \int_{t_1}^{t_1+t_2} \psi_0(t; z)dt = \int_0^{t_2} \log \frac{\psi(t_1, t_2'; z)}{\psi(t_1, 0; z)\psi(0, t_2'; z)} dt_2' + \int_0^{t_2} \psi_0(t; z)dt + t_2\ \psi_0(t_1; z).$$

Equating the relations (3.8) and (3.10), we get that

$$\text{(3. 11)} \qquad t_1\ \psi_0(t_2; z) - t_2\ \psi_0(t_1; z) = \int_0^{t_2} \log \frac{\psi(t_1, t_2'; z)}{\psi(t_1, 0; z)\psi(0, t_2'; z)} dt_2' - \int_0^{t_1} \log \frac{\psi(t_1', t_2; z)}{\psi(t_1', 0; z)\psi(0, t_2; z)} dt_1'$$

for $-\infty < t_1, t_2 < \infty$ and for all z. Dividing both sides of the equation by $t_1 t_2 \neq 0$, we have

$$\text{(3. 12)} \qquad \frac{\psi_0(t_2; z)}{t_2} - \frac{\psi_0(t_1; z)}{t_1} = \frac{1}{t_1 t_2}\left[\int_0^{t_2} \log \frac{\psi(t_1, t_2'; z)}{\psi(t_1, 0; z)\psi(0, t_2'; z)} dt_2' - \int_0^{t_1} \log \frac{\psi(t_1', t_2; z)}{\psi(t_1', 0)\psi(0, t_2; z)} dt_1'\right]$$

for $-\infty < t_1, t_2 < \infty, t_1 t_2 \neq 0$. Let $t_2 = t$ and $t_1 \to 0$. Assume that $m_0(z) = E(X_0|Z = z) < \infty$ and that the interchange of limit and the integral sign is permitted in the following computations. Then, we have

$$\lim_{t\to 0} \frac{\psi_0(t;z)}{t} = i m_0(z) \tag{3. 13}$$

and, from equation (3.12), we have

$$\begin{aligned}
\frac{\psi_0(t;z)}{t} &= i\, m_0(z) + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\frac{\psi(t_1,v;z)}{\psi(t_1,0;z)\psi(0,v;z)}dv \\
&\quad -\frac{1}{t_1}\int_0^{t_1}\log\frac{\psi(u,t;z)}{\psi(u,0;z)\psi(0,t;z)}du] \\
&= i\, m_0(z) + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\frac{\psi(t_1,v;z)}{\psi(t_1,0;z)\psi(0,v;z)}dv] \\
&\quad -\log\frac{\psi(0,t;z)}{\psi(0,0;z)\psi(0,t;z)} \\
&= i\, m_0(z) + \frac{1}{t}\lim_{t_1\to 0}[\int_0^t \frac{1}{t_1}\log\frac{\psi(t_1,v;z)}{\psi(t_1,0;z)\psi(0,v;z)}dv] \\
&= i\, m_0(z) + \frac{1}{t}\int_0^t \frac{\partial}{\partial u}[\log\frac{\psi(u,v;z)}{\psi(u,0;z)\psi(0,v;z)}]|_{u=0}dv.
\end{aligned} \tag{3. 14}$$

Hence

$$\psi_0(t;z) = it\, m_0(z) + \int_0^t \frac{\partial}{\partial u}[\log\frac{\psi(u,v;z)}{\psi(u,0;z)\psi(0,v;z)}]|_{u=0}dv. \tag{3. 15}$$

Using this explicit formula for $\psi_0(t;z)$, it is possible to compute $\phi_0(t;z)$ and hence compute $\phi_1(t;z)$ and $\phi_2(t;z)$ by using the relations

$$\phi_1(t;z) = \frac{\psi(t,0;z)}{\phi_0(t;z)}, \phi_2(t,z) = \frac{\psi(0,t;z)}{\phi_0(t;z)}, -\infty < t < \infty. \tag{3. 16}$$

Equations (3.15) and (3.16) give the explicit formulae for computing the characteristic functions of the conditional distributions of X_0, X_1 and X_2 given $Z = z$ provided the conditional characteristic function of $(X_0 + X_1, X_0 + X_2)$ given $Z = z$ is known and non-vanishing.

Remarks : (i) The assumption of the non-vanishing property of the conditional characteristic function of the bivariate random vector (Y_1, Y_2) given the random variable Z cannot be relaxed. This can be seen from the Example 2.1.1 in Prakasa Rao (1992). However, if the conditional characteristic functions of X_1, X_2, X_3 given the random variable Z are analytic, then Theorem 2.1 holds without the assumption of non-vanishing of the conditional characteristic functions. See Remark 2.1.5 in Prakasa Rao (1992).

(ii) Theorem 2.1 can be extended to n conditionally independent random variables. Suppose $X_i, 1 \leq i \leq n$ are conditionally independent random variables given a random variable Z. Let $Y_i = X_1 - X_n, 1 \leq i \leq n-1$. Suppose the conditional characteristic function of the vector $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})$ does not vanish. then the conditional joint distribution of $\mathbf{Y}$ given $Z = z$ determines the conditional distributions of $X_1, X_2, \ldots, X_n$ given $Z = z$ up to a change in location depending on z.

(iii) Theorem 2.1 can also be rephrased in terms of ratios instead of sums. Suppose X_1, X_2, X_3 are three conditionally independent positive random variables given a random variable Z. Let $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = \frac{X_2}{X_3}$. If the conditional characteristic function of $(\log Y_1, \log Y_2)$ given $Z = z$ does not vanish, then the conditional distribution of (Y_1, Y_2) given $Z = z$ determines the conditional distributions of X_1, X_2, X_3 up to a change of scale depending on z.

4 Identification of component distributions from joint distribution of maxima

Let X_0, X_1 and X_2 be conditionally independent random variables given a random variable Z Define $Y_1 = \max(X_0, X_1)$ and $Y_2 = \max(X_0, X_2)$.

Theorem 4.1: The conditional joint distribution of the vector (Y_1, Y_2) given the event $Z = z$ uniquely determines the conditional distributions of the random variables X_0, X_1 and X_2 given $Z = z$. provided the supports of the conditional distributions of X_0, X_1 and X_2 are the same given $Z = z$.

Proof : Let $F_i(x; z), i = 0, 1, 2$ and $F_i^*(x; z)$ denote alternate possibilities for the conditional distribution functions of X_i given $Z = z$ for $i = 0, 1, 2$. Let the conditional joint distribution of (Y_1, Y_2) given $z = z$ be denoted by $G(y_1, y_2; z)$. Then, for $-\infty < y_1 \leq y_2 < \infty$,

$$
\begin{aligned}
(4.\ 1) \qquad G(y_1, y_2; z) &= P(Y_1 \leq y_1, Y_2 \leq y_2 | Z = z) \\
&= P(X_0 \leq y_1, X_1 \leq y_1, X_0 \leq y_2, X_2 \leq y_2 | Z = z) \\
&= P(X_0 \leq y_1, X_1 \leq y_1, X_2 \leq y_2 | Z = z) \\
&= F_0(y_1; z) F_1(y_1; z) F_2(y_2; z)
\end{aligned}
$$

by the conditional independence of the random variables X_0, X_1 and X_2 given $Z = z$. Since $F_i*(.;z)$ is the alternate possibile distribution for the conditional distribution of $X_i, i = 0,1,2$ given $Z = z$, it follows that

$$F_0(y_1;z)F_1(y_1;z)F_2(y_2;z) = F_0^*(y_1;z)F_1^*(y_1;z)F_2^*(y_2;z) \tag{4. 2}$$

for $-\infty < y_1 \leq y_2 < \infty$ and for all z. Let $y_2 \to \infty$. Then it follows that

$$F_0(y_1;z)F_1(y_1;z) = F_0^*(y_1;z)F_1^*(y_1;z). -\infty < y_1 < \infty. \tag{4. 3}$$

Relations (4.2) and (4.3) show that

$$F_2(y_2;z) = F_2^*(y_2;z), -\infty < y_2 < \infty \tag{4. 4}$$

provided $F_0(y_1;z)F_1(y_1;z) > 0$. Note that, for any given z, the support of the function $F_0(.;z)F_1(.;z)$ is the same as the support of the function $F_0^*(.;z)F_1^*(.;z)$ from the equation (4.3). Let us now choose $-\infty < y_2 \leq y_1 < \infty$. Then

$$\begin{aligned} G(y_1,y_2;z) &= P(Y_1 \leq y_1, Y_2 \leq y_2 | Z = z) \\ &= P(X_0 \leq y_1, X_1 \leq y_1, X_0 \leq y_2, X_2 \leq y_2 | Z = z) \\ &= P(X_0 \leq y_2, X_1 \leq y_1, X_2 \leq y_2 | Z = z) \\ &= F_0(y_2;z)F_1(y_1;z)F_2(y_2;z) \end{aligned} \tag{4. 5}$$

by the conditional independence of the random variables X_0, X_1 and X_2 given $Z = z$. This relation gives the equation

$$F_0(y_2;z)F_1(y_1;z)F_2(y_2;z) = F_0^*(y_2;z)F_1^*(y_1;z)F_2^*(y_2;z) \tag{4. 6}$$

for $-\infty < y_2 \leq y_1 < \infty$ and for all z. Let $y_1 \to \infty$. Then it follows that

$$F_0(y_2;z)F_2(y_2;z) = F_0^*(y_2;z)F_2^*(y_2;z), -\infty < y_2 < \infty. \tag{4. 7}$$

Equations (4.6) and (4.7) show that

$$F_1(y_1;z) = F_1^*(y_1;z), -\infty < y_1 < \infty \tag{4. 8}$$

provided $F_0(y_2;z)F_2(y_2;z) > 0$. Note that the support of the function $F_0(.;z)F_2(.;z)$ is the same as the support of the function $F_0^*(.;z)F_2^*(.;z)$ from the equation (4.6). Since the supports of the conditional distribution functions $F_0(.;z), F_1(.;z), F_2(.;z)$ given $Z = z$ are the same, it follows that

$$F_i(y;z) = F_i^*(y;z) \tag{4. 9}$$

from the equations (4.2), (4.4) and (4.8) over the common support of X_0, X_1 and X_2 given $Z = z$. Hence the conditional distribution of (Y_1, Y_2) given $Z = z$, uniquely determines the conditional distributions of X_0, X_1 and X_2 given $Z = z$.

Remarks : (i) It is known that the distribution of the random variable $Y_1 = \max(X_0, X_1)$ alone cannot determine the distributions of X_0 and X_1 uniquely even if X_1 and X_2 are independent unless X_0 and X_1 are independent and identically distributed random variables. A similar observation holds for the conditional distribution of Y_1 given $Z = z$.

(ii) Given the conditional joint distribution $G(y_1, y_2; z)$ of the bivariate random vector (Y_1, Y_2) given $Z = z$ in Theorem 4.1, it is possible to explicitly determine the conditional distributions $F_i(.;z), i = 0, 1, 2$ and they are given by

$$F_0(x;z) = \frac{G(x,\infty;z)G(\infty,x;z)}{G(x,x;z)}, \tag{4. 10}$$

$$F_1(x;z) = \frac{G(x,x;z)}{G(\infty,x;z)}, \tag{4. 11}$$

and

$$F_2(x;z) = \frac{G(x,x;z)}{G(x,\infty;z)}. \tag{4. 12}$$

This can be checked by using the relation (4.5) and following methods in Kotlarski (1978).

(iii) A result similar to Theorem 4.1 can be proved for minima of random variables following Theorem 2.3.1 in Prakasa Rao (1992).

5 Identification of component distributions from joint distribution of Maximum and Minimum

Let X_0, X_1, X_2 be conditionally independent random variables given a random variable Z. Let $Y_1 = \min(X_0, X_1)$ and $Y_2 = \max(X_0, X_2)$.

Theorem 5.1: Let $F_i(.;z)$ be the conditional distribution X_i given $Z = z$ for $i = 0, 1, 2$. Suppose that, for some fixed a, b, x_0, q satisfying $-\infty \leq a < x_0 < b \leq \infty, 0 < q < 1$ possibly depending on z,

(5. 1) $$F_1(x;z) < 1, x < b\,; F_1(b-0;z) = 1 \text{ if } b < \infty,$$

(5. 2) $$F_2(y;z) < 1, y > a\,; F_(a+0;z) = 0 \text{ if } a > -\infty,$$

(5. 3) $$F_0(a+0;z) = 0, F_0(b-0;z) = 1, F_0(x_0;z) = q$$

and $F_0(.;z)$ is strictly increasing in (a,b). Then the conditional joint distribution of (Y_1, Y_2) given $Z = z$ uniquely determines the conditional distributions of $F_0(.;z), F_1(.;z)$ and $F_2(.;z)$.

Proof : Let y_1 and y_2 be chosen such that $-\infty < y_1 \leq y_2 < \infty$. Then

(5. 4) $$\begin{aligned} P(Y_1 > y_1, Y_2 \leq y_2 | Z = z) &= P(X_0 > y_1, X_1 > y_1, X_0 \leq y_2, X_2 \leq y_2 | Z = z) \\ &= P(y_1 < X_0 \leq y_2, X_1 > y_1, X_2 \leq y_2)|Z = z) \\ &= (F_0(y_2;z) - F_0(y_1;z))\bar{F}_1(y_1;z)F_2(y_2;z) \end{aligned}$$

where $\bar{F}_i(y;z) = 1 - F_i(y;z), i = 0,1,2$. Suppose that $\{F_0^*(.;z), F_1^*(.;z), F_2^*(.;z)\}$ is another set of conditional distributions for $\{X_0, X_1, X_2\}$ given $Z = z$ satisfying the conditions stated in the theorem such that the conditional distributions of (Y_1, Y_2) given $Z = z$ are the same under $\{F_i(.;z), i = 0,1,2\}$ as well as $\{F_i^*(.;z), i = 0,1,2\}$. Then

(5. 5) $$\begin{aligned} &(F_0^*(y_2;z) - F_0^*(y_1;z))\bar{F}_1^*(y_1;z)F_2^*(y_2;z) \\ &= (F_0(y_2;z) - F_0(y_1;z))\bar{F}_1(y_1;z)F_2(y_2;z) \end{aligned}$$

for $-\infty < y_1 \leq y_2 < \infty$ and for all z. Let $y_2 \to \infty$ in (5.5). Then

(5. 6) $$\bar{F_0^*}(y_1;z))\bar{F_1^*}(y_1;z) = \bar{F}_0(y_1;z))\bar{F}_1(y_1;z), -\infty < y_1 < \infty$$

for all z. Let $y_1 \to \infty$ in (5.5). We get that

(5. 7) $$F_0^*(y_2;z)F_2^*(y_2;z) = F_0(y_1;z))F_2(y_2;z), -\infty < y_2 < \infty$$

for all z. Combining the relations (5.5) to (5.7), we get that

(5. 8)

$$\begin{aligned} &(F_0^*(y_2;z) - F_0^*(y_1;z))\bar{F}_1^*(y_1;z)F_2^*(y_2;z)\bar{F}_0(y_1;z))\bar{F}_1(y_1;z)F_0(y_1;z))F_2(y_2;z) \\ &= (F_0(y_2;z) - F_0(y_1;z))\bar{F}_1(y_1;z)F_2(y_2;z)\bar{F_0^*}(y_1;z))\bar{F_1^*}(y_1;z)F_0^*(y_2;z))F_2^*(y_2;z) \end{aligned}$$

for $-\infty < y_1 \leq y_2 < \infty$ and for all z. Applying the conditions (5.1) to (5.3), we have

(5. 9) $$\frac{F_0^*(y_2;z) - F_0^*(y_1;z)}{F_0(y_2;z) - F_0(y_1;z)} = \frac{\bar{F}_0^*(y_1)}{\bar{F}_0(y_1)} \frac{F_0^*(y_1)}{F_0(y_1)}$$

for $-\infty \leq a < y_1 < y_2 < b \leq \infty$ and for all z. Since $F_0^*(x_0;z) = F_0(x_0;z) = q$, it follows that, for $-\infty \leq a < y \leq x_0$,

(5. 10) $$\frac{F_0^*(x_0;z) - F_0^*(y;z)}{F_0(x_0;z) - F_0(y;z)} = \frac{\bar{F}_0^*(y)}{\bar{F}_0(y)}.$$

Hence

(5. 11) $$F_0^*(y;z) = F_0(y;z), -\infty < y \leq x_0$$

for all$z$. Similar arguments show that

(5. 12) $$F_0^*(y;z) = F_0(y;z), x_0 \leq y < \infty.$$

for all z. Equations (5.6) and (5.7) prove that

(5. 13) $$F_1^*(y;z) = F_1(y;z) \text{ and } F_2^*(y;z) = F_2(y;z), -\infty < y < \infty$$

for all z.

Remarks : (i) Given the conditional joint distribution of (Y_1, Y_2) with $Y_1 = \min(X_0, X_1)$ and $Y_2 = \max(X_0, X_2)$ given $Z = z$, one can explicitly compute the conditional distributions of X_0, X_1, X_2 given $Z = z$, following the methods in Kotlarski (1978). Let

(5. 14) $$\begin{aligned} H(u,v;z) &= P(Y_1 > u, Y_2 \leq v | Z = z) \\ &= \bar{F}_1(u;z) F_2(v;z)[F_0(v;z) - F_0(u;z)] \end{aligned}$$

for $-\infty < u < v < \infty$. It can be shown that

(5. 15) $$\begin{aligned} F_0(x;z) &= \frac{q[H(x,x_0;z) - H(-\infty,x_0;z)H(x,\infty;z)]}{qH(x,x_0;z) - H(-\infty,x_0;z)H(x,\infty;z)} \text{ if } x \leq x_0 \\ &= \frac{qH(x_0,\infty;z)H(-\infty,x;z)}{H(x_0,\infty;z)H(-\infty,x;z) - (1-q)H(x_0,x;z)} \text{ if } x \geq x_0, \end{aligned}$$

(5. 16) $$\bar{F}_1(x;z) = \frac{H(x,\infty;z)}{\bar{F}_0(x;z)}, -\infty < x < \infty$$

and

(5. 17) $$F_2(y;z) = \frac{H(-\infty,y;z)}{F_0(y;z)}, -\infty < y < \infty,$$

for all z where x_0 and q are as defined in Theorem 5.1. These results follow by methods in Kotlarski (1978).

(ii) Following results in Kotlarski (1978) and Prakasa Rao (1992), it is possible to obtain other results similar to those in Sections 2 to 5 based on identifiability from product and minimum (or maximum) or identifiability from products and sums or identifiability from sum and maximum (or minimum).

6 Identifiability by maxima of several random variables

Let $X_1, X_2, \ldots, X_n$ be conditionally independent positive random variables given a random variable Z. Let $F_i(x; z)$ be the distribution function of X_i given $Z = z$ for $i = 1, \ldots, n$. Suppose that $F_i(x; z) > 0$ for all $x > 0$ and for all z for $i = 1, \ldots, n$. Define

$$\begin{aligned} Y_1 &= \max(a_1 X_1, \ldots, a_n X_n) \\ Y_2 &= \max(b_1 X_1, \ldots, b_n X_n) \end{aligned} \tag{6. 1}$$

where $a_i > 0, b_i > 0$ for $i = 1, \ldots, n$ and $\frac{a_i}{b_i} \neq \frac{a_j}{b_j}$ for $1 \leq i \neq j \leq n$.

Theorem 6.1 : Under the conditions stated above, the conditional joint distribution of (Y_1, Y_2) given $Z = z$ uniquely determines the conditional distribution of the random variable X_i given $Z = z$ for $1 \leq i \leq n$.

Proof: Let $F_j^*(.; z)$ be an alternate possible conditional distribution of X_j given $Z = z$ for $1 \leq j \leq n$. Note that

$$\begin{aligned} H(t, s; z) &\equiv P(Y_1 \leq t, Y_2 \leq s | Z = z) \\ &= \Pi_{j=1}^n F_j(\min(\frac{t}{a_j}, \frac{s}{b_j}); z) \end{aligned} \tag{6. 2}$$

for $0 \leq t, s < \infty$ and for all z. Since $F_j^*(.; z)$ is an alternate possible conditional distribution of X_j for $1 \leq j \leq n$, it follows that

$$\Pi_{j=1}^n F_j(\min(\frac{t}{a_j}, \frac{s}{b_j}); z) = \Pi_{j=1}^n F_j^*(\min(\frac{t}{a_j}, \frac{s}{b_j}); z) \tag{6. 3}$$

for $0 \leq t, s < \infty$ and for all z. Let $v_j(t;z) = \log F_j(\frac{t}{b_j};z) - \log F_j^*(\frac{t}{b_j};z)$. The equation (6.3) can be written in the form

$$\sum_{j=1}^{n} v_j(\min(c_j t, s); z) = 0, 0 \leq t, s < \infty \tag{6. 4}$$

for all z where $c_j = \frac{b_j}{a_j}, 1 \leq j \leq n$ are pairwise distinct. Without loss of generality, assume that $0 < c_1 < \ldots < c_n$. Let $t > 0$ and $s = \tau t$ where $c_{n-1} < \tau < c_n$. Then the equation (6.4) can be written in the form

$$\sum_{j=1}^{n-1} v_j(c_j t; z) + v_n(\tau t; z) = 0, 0 < t < \infty \tag{6. 5}$$

for all z. This equation shows that $v_n(,;z)$ is a constant depending only on z on the interval $(c_{n-1}t, c_n t)$ for any $t > 0$. Since $t > 0$ is arbitrary, it follows that $v_n(.;z)$ is constant depending only on z on the interval $(0, \infty)$. Since $v_j(t;z) \to 0$ as $t \to \infty$, it follows that $v_n(t;z) = 0$ for $t > 0$ for all z. Repeating this argument, it is easy to see that

$$v_j(t;z) = 0, 1 \leq j \leq n-1, 0 < t < \infty \tag{6. 6}$$

for all z. This in turn implies that

$$F_j(\frac{t}{b_j};z) = F_j^*(\frac{t}{b_j};z), 0 < t < \infty, 1 \leq j \leq n \tag{6. 7}$$

from the definition of $v_j(.;z)$. Since $t > 0$ is arbitrary, it follows that

$$F_j(t;z) = F_j^*(t;z), 0 < t < \infty, 1 \leq j \leq n. \tag{6. 8}$$

Remarks : The result stated in Theorem 6.1 does not hold for random variables $X_i, 1 \leq i \leq n$ taking positive and negative values with positive probability. This can be seen by the Example 2.8.1 in Prakasa Rao (1992). However the following result holds.

Theorem 6.2 : Suppose that the random variables $X_i, 1 \leq i \leq n$ are conditionally independent given a random variable Z. Let $F_i(x;z)$ be the conditional distribution function of X_i given $Z = z$, for $1 \leq i \leq n$. Further suppose that $F_j(x;z) > 0, 1 \leq j \leq n$ for all $x \in R$ and for all z and $P(X_j = 0|Z = z) = 0, 1 \leq j \leq n$. Define

$$\begin{aligned} Y_1 &= \max(a_1 X_1, \ldots, a_n X_n) \\ Y_2 &= \max(b_1 X_1, \ldots, b_n X_n) \end{aligned} \tag{6. 9}$$

where $a_i > 0, b_i > 0$ for $i = 1, \ldots, n$ and $\frac{a_i}{b_i} \neq \frac{a_j}{b_j}$ for $1 \leq i \neq j \leq n$. Then the conditional joint distribution of (Y_1, Y_2) given $Z = z$ uniquely determines the conditional distribution of the random variable X_i given $Z = z$ for $1 \leq i \leq n$.

Proof: Following the arguments in the proof of Theorem 6.1, we get that

$$\sum_{j=1}^{n} v_j(\min(c_j t, s); z) = 0, -\infty, t, s < \infty \tag{6. 10}$$

for all z where $c_j = \frac{a_j}{b_j}$ are pairwise distinct and $0 < c_1 < \ldots < c_n$. Following the same arguments again, it follows that $v_j(t; z) = 0, t > 0$ for all z. Suppose that $t < 0$. Let $s = \tau t, \tau \in (c_1, c_2)$. Then the equation (6.4) takes the form

$$v_1(\tau t; z) + \sum_{j=2}^{n} v_j(c_j t; z) = 0 \tag{6. 11}$$

for all z. Hence $v_1(.; z)$ is a constant depending only on z on the interval $(c_2 t, c_1 t)$. Since $t < 0$ is arbitrary, it follows that $v_(t; z) = 0$ on the interval $(-\infty, 0)$. Note that $v_1(x; z)$ is continuous at $x = 0$. Hence $v_1(0; z) = 0$ for all z. Therefore $v_(1t; z) = 0$ for all t and for all z. Combining with earlier remarks, we get that $v_1(t; z) = 0$ for all t and for all z. By repeating the arguments recursively, we get that $v_j(t; z) = 0, -\infty < t < \infty, 1 \leq j \leq n$ for all z. Hence $F_j(t; z) = F_j^*(t; z), -\infty < t < \infty, 1 \leq j \leq n$ for all z.

Proofs of Theorems 6.1 and 6.2 are akin to those in Klebanov (1973) in the independent case.

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