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Parameter Estimation for 2D Stochastic Navier-Stokes Equation Driven by Infinite Dimensional Fractional Brownian motion

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Abstract

We study parameter estimation for a two-dimensional stochastic Navier-Stokes equation driven by an infinite dimensioonal fractional Brownian motion.

Key Words :Parameter estimation, Stochastic Navier-Stokes equation, Maximum likelihood estimation, fractional Brownian motion.

AMS Subject Classification (2000) : Primary 62M40.

1 Introduction

Long range dependence phenomena is said to occur in a time series $\{X_n, n \geq 0\}$ if the $Cov(X_0, X_n)$ of the time series tends to zero as $n \to \infty$ and yet it satisfies the condition

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| = \infty.$$

In other words, $Cov(X_0, X_n)$ tends to zero but so slowly that their sums diverge. This phenomenon was first observed by the hydrologist Hurst (1951) on projects involving the design of reservoirs along the Nile river (cf. Montanari (2003)) and by others in hydrological time series. It was observed that a similar phenomenon occurs in problems concerned with modelling traffic patterns of packet flows in high-speed data net works such as internet (cf. Willinger et al. (2003), Norros (2003)). Long range dependence is related to the concept of self-similarity for a stochastic process. A stochastic process $\{X(t), t \in R\}$ is said to be H-self-similar with index $H > 0$, if for every $a > 0$, the process $\{X(at), t \in R\}$ and the process $\{a^H X(t), t \in R\}$ have the same finite dimensional distributions. If a process is self-similar with stationary increments, then the increments form a stationary time series exhibiting long range dependence. A gaussian self-similar process with stationary increments $0 < H < 1$ is called a fractional Brownian motion (fBm).

Diffusion processes and diffusion type processes satisfying stochastic differential equations (SDE) are used for stochastic modeling in a wide variety of sciences such as population genetics, economic proceeses, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1.1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Maximum likelihood estimation and Bayes estimation for more general classes of stochastic processes satisfying linear stochastic differential equations driven fractional Brownian motion is investigated in Prakasa Rao (2003). A comprehensive review on statistical inference for fractional diffusion processes is given in a monograph in Prakasa Rao (2010).

Parameter estimation for the stochastically perturbed Navier-Stokes equations has been recently studied by Cialenco and Glatt-Hotz (2011). They consider a parameter estimation problem to determine the viscosity ν of a stochastically perturbed 2D Navier-Stokes system driven by Brownian motion. They derive different types of estimators based on a single sample path observed over a finite time interval and showed that these estimators are consistent and asymptotically normal under some conditions. We have studied problems of parameter estimation for stochastic partial differential equations driven by an infinite dimensional fractional Brownian motion in Prakasa Rao (2004).

Our aim in this paper is to study the problems of parameter estimation for two-dimensional stochastically perturbed Navier-Stokes equations driven by an infinite dimensional fractional Brownian motion based on a single sample path observed over a finite time interval.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions.The natural fitration of a stochastic process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2.1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0 \tag{2.2}$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0 \tag{2.3}$$

driven by the fractional Brownian motion W^H. The integral

$$\int_0^t B(s)dW_s^H \tag{2.4}$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBM in a natural sense (cf. Norros et al. (1999)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2.5}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2.6}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2.7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.8}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2.9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variation $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBM W^H. In fact the stochastic integral

$$\int_0^t B(s)dW_s^H \tag{2.10}$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr, 0 \leq s \leq t \tag{2.11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Therorem 2.1: Let M^H be the fundamental martingale associated with the fBM W^H defined by (2.9). Then

$$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, t \in [0,T] \tag{2.12}$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. (1993)) so that

$$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}ds, t \in [0,T] \tag{2.13}$$

is well-defined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absoulute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by

$$Z_t = \int_0^t k_H(t,s)B^{-1}(s)dY_s \tag{2.14}$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold:

(i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s)dw_s^H + M_t^H \tag{2.15}$$

where M^H is the fundamental martingale defined by (2.9),
(ii) the process Y admits the representation

$$Y_t = \int_0^t K_H^B(t,s)dZ_s \tag{2.16}$$

where the function K_H^B is as defined in (2.11), and

(iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000) derived the following Girsanov type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}. \tag{2.17}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t B(s) dW_s^H, 0 \leq t \leq T. \tag{2.18}$$

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3 Stochastic 2D Navier-Stokes equation driven by an infinite dimensional fBm

Consider the 2D Navier-Stokes equation driven by an infinite dimensional fBM with Hurst index H:

$$\begin{aligned} dU + ((U.\bigtriangledown)U - \nu \bigtriangleup U + \bigtriangledown P)dt &= \sigma dW^H, \\ \bigtriangledown.U &= 0, \\ U(0) &= U_0 \end{aligned} \tag{3.1}$$

which models the flow of a viscous incompressible fluid. Here $U = (U_1, U_2)$ is the velocity field and P the pressure. The parameter $\nu > 0$ is the kinematic viscosity of the fluid and is the parameter to be estimated. We would like to estimate the parameter ν based on the sample path $U(\omega)$ observed over a finite time interval $[0, T]$. We assume that the governing equations (3.1) evolve over a domain D. We will consider two possible boundary conditions. In the first case, we suppose that the flow occurs over all of R^2 with $D = [-L/2, L/2]^2$ for some $L > 0$ and we impose the *periodic boundary condition*:

$$U(\mathbf{x} + L\mathbf{e}_j, t) = U(\mathbf{x}, t), \mathbf{x} \in R^2, t \geq 0; \int_D U(\mathbf{x}) d\mathbf{x} = 0. \tag{3.2}$$

We will also consider the case when D is a bounded subset of R^2 with a smooth boundary ∂D and assume the *Dirichlet (no slip) boundary condition*:

$$U(\mathbf{x}, t) = 0, x \in \partial D, t \geq 0. \tag{3.3}$$

The stochastic forcing function is an additive space-time fractional noise given by

$$\sigma dW^H = \sum_k \lambda_k^{-\gamma} \Phi_k dW_k^H \tag{3.4}$$

where Φ_k are the eigenfunctions of the Stokes operator, λ_k are the associated eigenvalues and $W_k^H, k \geq 1$ are one-dimensional independent fractional Brownian motions. We assume that γ is real constant greater than one. Note that the space-time correlation structure can be written formally as

$$E[\sigma dW^H(\mathbf{x},t)\sigma dW^H(\mathbf{y},s)] = K(\mathbf{x},\mathbf{y})(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \tag{3.5}$$

where

$$K(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{\infty} \lambda_k^{-2\gamma} \Phi_k(\mathbf{x})\Phi_k(\mathbf{y}).$$

Consider the projection of the equation (3.1) down to a finite dimensional space and for each N we consider a stochastic system of the form

$$dU^N + (\nu AU^N + P_N B(U))dt = P_N \sigma dW^H;\ U(0) = U_0 \tag{3.6}$$

where P_N is the projection operator on the finite dimensional space generated by the first N Fourier eigenvalues of the Stokes operator.

We now describe the mathematical background for the stochastic Navier-Stokes equations following Cialenco and Glatt-Holtz (2011). We can formulate the equation (3.1) as an infinite dimensional stochastic evolution equation of the form

$$\begin{aligned} dU + (\nu AU + B(U))dt &= \sigma dW^H \\ U(0) &= U_0. \end{aligned} \tag{3.7}$$

We now introduce basic function spaces in detail as described in Cialenco and Glatt-Holtz (2011) for completeness.

Let us first consider the space associated with a Dirichlet (no slip) boundary condition. Let $J = \{U \in L^2(D)^2 : \bigtriangledown.U = 0, U.n = 0\}$ where n is the outer pointing unit normal to ∂D. The set J is endowed as a Hilbert space with the L^2 inner product $(U^b, U^c) = \int_D U^b U^c dx$ and the associated norm $|U| = (U,U)^{1/2}$. The Lerang-Hopf projector P_J is defined as the orthogonal projection of $L^2(D)^d$ onto J. Let $V = \{U \in H_0^1(D)^2 : \bigtriangledown.U = 0\}$ and endow this space with the inner product $((U^b, U^c)) = \int_M \bigtriangledown U^b. \bigtriangledown U^c dx$. Due to the Dirichlet boundary condition (3.3), the Poincare inequality $|U| \leq c||U||$ holds for $U \in V$ justifying this definition.

Suppose the periodic boundary condition (3.2) holds. We take $D = [-L/2, L/2]^2$ and define the spaces $L^2_{per}(D)^2, H^1_{per}(D)^2$ to be the families of vector fields $U = U(\mathbf{x})$ which are

L periodic in each direction and which belong to $L^2(G)^2$ and $H^1(G)^2$ respectively for every open bounded set $G \subset R^2$.. We define

$$J = \{U \in L^2_{per}(D)^2 : \bigtriangledown .U = 0, \int_D U(\mathbf{x})d\mathbf{x} = 0\} \tag{3.8}$$

and

$$V = \{U \in H^1_{per}(D)^2 : \bigtriangledown .U = 0, \int_D U(\mathbf{x})d\mathbf{x} = 0\}. \tag{3.9}$$

The spaces J and V are endowed with the norms $|.|$ and $||.||$ respectively as defined earlier. We impose the mean zero condition for defining J and V so that the Poincare inequality holds (cf. Temam (1995)).

The linear portion of the equation (3.1) is described by the Stokes operator $A = -P_J \triangle$ which is an unbounded operator from J to J with the domain $D(A) = H^2(M) \cap V$. Since A is self-adjoint with a compact inverse $A^{-1} : J \to D(A)$, we apply the theory of compact symmetric operators which ensure the existence of an orthonormal basis $\{\Phi_k, k \geq 1\}$ for J of eigenfunctions of A with the associated eigenvalues $\{\lambda_k, k \geq 1\}$ forming an unbounded increasing sequence. Furthermore

$$\frac{\lambda_k}{k\lambda_1} \to 1 \text{ as } k \to \infty.$$

For more details on the asymptotic behaviour of the sequence $\{\lambda_k, k \geq 1\}$, see Babenko (1982) and Metivier (1978) for the no slip case (3.3) and Constantin and Foias (1988) for the periodic case (3.2).

Define $H_N = \text{Span}\{\Phi_1, \ldots, \Phi_N\}$ and let P_N be the projection from J onto this space. We let $\tilde{P}_N = I - P_N$.

Given $\alpha > 0$, let $D(A^\alpha) = \{U \in J : \sum_k \lambda_k^{2\alpha} |u_k|^2 < \infty\}$ where $u_k = (U, \Phi_k)$. For $U \in D(A^\alpha)$, define

$$A^\alpha U = \sum_k \lambda_k^\alpha u_k \Phi_k$$

whenever $U = \sum_k u_k \Phi_k$. Note that

$$|A^{\alpha_2} P_N U| \leq \lambda_N^{\alpha_2 - \alpha_1} |A^{\alpha_1} P_N U| \tag{3.10}$$

and

$$|A^{\alpha_1} \tilde{P}_N U| \leq \lambda_N^{\alpha_1 - \alpha_2} |A^{\alpha_2} \tilde{P}_N U| \tag{3.11}$$

for any $\alpha_1 < \alpha_2$.

We now consider the stochastic part of the equation (3.1). We fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P, \{W_k^H, k \geq 1\})$ which is a filtered probability space with $\{W_k^H, k \geq 1\}$, a sequence of independent standard fractional Brownian motions relative to the filtration $\{\mathcal{F}_t, t \geq 0\}$. We assume that $\mathcal{F}_t$ is complete and right continuous (cf. Da Prato and Zabczyk (1992)). Formally, $W^H = \sum_{k\geq 0} \Phi_k W_k^H$ and W^H can be considered as a cylindrical fractional Brownian motion.

Consider the family of Hilbert-Schmidt operators mapping J into $D(A^\beta), \beta \geq 0$. We denote the family by $L_2(H, D(A^\beta))$. We assume that σ, understood as an operator, has the form

$$\sigma \Phi_k = \lambda_k^{-\gamma} \Phi_k$$

and we write

$$\sigma dW^H(t) = \sum_{k=1}^{\infty} \lambda_k^{-\gamma} \Phi_k dW_k^H(t), t \geq 0.$$

4 Estimation for stochastic Stokes equation

Stochastic differential equations in infinite dimensions driven by a cylindrical Brownian motion are investigated in Rozovskii (1990), Da Prato and Zabczyk (1992) and Kallianpur and Xiong (1995) among others. Stochastic partial differential equations driven by infinite dimensional fractional Brownian motion have been studied in Tindel et al. (2003) and estimation for parameters involved for such processes are discussed in Prakasa Rao (2004, 2010).

We consider the linear system associated to (3.7):

$$d\bar{U} + \nu A\bar{U}dt = \sum_k \lambda_k^{-\gamma} \Phi_k dW_k^H, \;\; \bar{U}(0) = U_0. \tag{4.1}$$

Let $\{\tilde{u}_k, k \geq 1\}$ denote the Fourier coefficients of the solution $\bar{U}$ with respect to the system $\{\Phi_k, k \geq 1\}$ in J, that is $\tilde{u}_k = (\bar{U}, \Phi_k), k \geq 1$. By (4.1), we observe that each Fourier coefficient $\tilde{u}_k$ represents a one-dimensional fractional Ornstein-Uhlenbeck process satisfying the stochastic differential equation

$$d\tilde{u}_k + \nu\lambda_k \tilde{u}_k dt = \lambda_k^{-\gamma} dW_k^H, \tilde{u}_k(0) = \bar{u}_{0k}, k \geq 1. \tag{4.2}$$

Let Pr_ν be the probability measure generated by $\bar{U}$ when ν is the true parameter. Suppose ν_0 is the true parameter. Following the discussion in Section 2, we define

$$M_k^H(t) = \int_0^t k_H(t,s)dW_k^H(s), 0 \leq t \leq T, \tag{4.3}$$

$$Q_k(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\tilde{u}_k(s)ds, 0 \leq t \leq T \tag{4.4}$$

and

$$Z_k(t) = \int_0^t k_H(t,s)d\tilde{u}_k(s), 0 \leq t \leq T. \tag{4.5}$$

Then

$$Z_k(t) = -\nu\lambda_k \int_0^t Q_k(s)dw_s^H + \lambda_k^{-\gamma}M_k^H(t) \tag{4.6}$$

and it follows that

$$\tilde{u}_k(t) = \int_0^t K_H(t,s)dZ_k(s) \tag{4.7}$$

where

$$K_H(t,s) = H(2H-1)\frac{d}{ds}\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, 0 \leq s \leq t. \tag{4.8}$$

Then M_k^H is a zero mean Gaussian martingale. Furthermore, it follows that the process $\{Z_k(t), t \geq 0\}$ is a semimartingale and the natural filtrations $(\mathcal{Z}_{k,t})$ and $(\mathcal{U}_{k,t})$ of the processes $\{Z_k(t), t \geq 0\}$ and $\{\tilde{u}_k(t), t \geq 0\}$ respectively coincide. Let $Pr_{k,\nu}$ be the probability measure generated by the process $\{\tilde{u}_k(t), 0 \leq t \leq T\}$ when ν is the true parameter. Let ν_0 be the true parameter. It follows by the Girsanov type theorem discussed in Section 2 that

$$\begin{aligned} \log\frac{dPr_{k,\nu}}{dPr_{k,\nu_0}} &= (-\nu\lambda_k + \nu_0\lambda_k)\int_0^T Q_k(t)dZ_k(t) \\ &\quad -\frac{1}{2}((-\nu\lambda_k)^2 - (-\nu_0\lambda_k)^2)\int_0^T Q_k^2(t)dw_t^H. \end{aligned} \tag{4.9}$$

Let

$$u^N(t,x) = \sum_{k=1}^N \tilde{u}_k(t)\Phi_k(x) \tag{4.10}$$

From the independence of the processes $\{W_i^H, 1 \leq i \leq N\}$ and hence of the processes $\{\tilde{u}_k, 1 \leq k \leq N\}$, it follows that the Radon-Nikodym derivative of the probability measure Pr_ν^N generated by the process $\{u^N(t), 0 \leq t \leq T\}$ when ν is the true parameter with respect to the probability measure $Pr_{\nu_0}^N$ generated by the process $\{u^N(t), 0 \le t \le T\}$ when ν_0 is the true parameter is given by

$$\log \frac{dPr_{\nu}^N}{dPr_{\nu_0}^N}(u^N) = \sum_{k=1}^{N}[\lambda_k(-\nu+\nu_0)\int_0^T Q_k(t)dZ_k(t) - \frac{1}{2}\lambda_k^2(\nu^2-\nu_0^2)\int_0^T Q_k^2(t)dw_t^H]. \tag{4.11}$$

It is easy to check that the maximum likelihood estimator $\hat{\nu}_N$ of the parameter ν based on the projection u^N of u is given by

$$\hat{\nu}_N = -\frac{\sum_{k=1}^N \lambda_k \int_0^T Q_k(t)dZ_k(t)}{\sum_{k=1}^N \lambda_k^2 \int_0^T Q_k^2(t)dw_t^H}. \tag{4.12}$$

Suppose ν_0 is the true parameter. Then

$$\hat{\nu}_N - \nu_0 = \frac{\sum_{k=1}^N \lambda_k^{-\gamma+1} \int_0^T Q_k(t)dM_k^H(t)}{\sum_{k=1}^N \lambda_k^{-2\gamma+2} \int_0^T Q_k^2(t)dw_t^H}. \tag{4.13}$$

Observe that the processes $M_k^H, 1 \le k \le N$ are independent zero mean Gaussian martingales with $< M_k^H >= w^H, 1 \le k \le N$.

We will now state two results giving the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) for sums of independent random variables which will be used to prove the consistency and the asymptotic normality of the maximum likelihood estimator $\hat{\nu}_N$ under some conditions.

Theorem 4.1 : (LLN) Let $\{\psi_n, n \ge 1\}$ be a sequence of independent random variables with finite variances and $\{b_n, n \ge 1\}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. Further suppose that

$$\sum_{n=1}^{\infty} \frac{Var(\psi_n)}{b_n^2} < \infty. \tag{4.14}$$

Then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (\psi_k - E[\psi_k])}{b_n} = 0 \text{ a.s.} \tag{4.15}$$

Theorem 4.2 : (CLT) Let $\mathcal{S} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \ge 0\}, \{M_k, k \ge 1\})$ be a stochastic basis where $M_k, k \ge 1$ is a sequence of independent martingales. Suppose further that R_k is a predictable process corresponding to the martingale M_k for each $k \geq 1$ such that

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \int_0^T R_k^2(t) d < M_k > (t)}{\sum_{k=1}^{N} E(\int_0^T R_k^2(t) d < M_k > (t))} = 1 \tag{4.16}$$

in probability. Then

$$\frac{\sum_{k=1}^{N} \int_0^T R_k(t) dM_k(t)}{[\sum_{k=1}^{N} \int_0^T E(R_k^2(t)) d < M_k(t) >]^{1/2}} \xrightarrow{\mathcal{L}} N(0,1) \text{ as } N \to \infty. \tag{4.17}$$

Theorem 4.1 follows from Theorem IV.3.2. in Shiryayev (1996) and Theorem 4.2 follows from Theorem 5.5.4 (II) in Liptser and Shiryayev (1989) or from related results in Prakasa Rao (1999).

Let

$$\alpha_k = \lambda_k^{-\gamma+1} \int_0^T Q_k(t) dM_k^H(t), k \geq 1 \tag{4.18}$$

and

$$\beta_k = \lambda_k^{-2\gamma+2} \int_0^T Q_k^2(t) dw_t^H, k \geq 1. \tag{4.19}$$

Observe that the sequence $\{\alpha_k, k \geq 1\}$ is a sequence of independent random variables with mean zero and

$$Var(\alpha_k) = \lambda_k^{-2\gamma+2} \int_0^T E(Q_k^2(t)) dw_t^H = E(\beta_k), k \geq 1. \tag{4.20}$$

and the sequence $\{\beta_k, k \geq 1\}$ is also a sequence of nonnegative independent random variables. Assume that $E(\beta_k) < \infty, k \geq 1$. Let

$$b_N = \sum_{k=1}^{N} Var(\alpha_k) = \sum_{k=1}^{N} E(\beta_k). \tag{4.21}$$

Note that $\{b_N, N \geq 1\}$ is an increasing sequence of positive numbers. Suppose the sequence $\{b_N, N \geq 1\}$ satisfies the following conditions:

(C1) $b_N \to \infty$ as $N \to \infty$ and $\sum_{N=1}^{\infty} \frac{Var(\alpha_k)}{b_k^2} < \infty$; and

(C2) $\frac{b_N}{\sum_{k=1}^{N} \beta_k} = O(1)$ a.s..

Then

$$\begin{aligned}(4.22)\qquad \hat{\nu}_N - \nu_0 &= \frac{\sum_{k=1}^N \alpha_k}{\sum_{k=1}^N \beta_k} \\ &= \frac{\sum_{k=1}^N \alpha_k}{\sum_{k=1}^N E(\beta_k)} \frac{b_N}{\sum_{k=1}^N \beta_k} \\ &= T_{1N}T_{2N} \ \text{(say)}.\end{aligned}$$

Under the condition $(C1)$, Theorem 4.1 implies that $T_{1,N} \to 0$ almost surely as $N \to \infty$ and $T_{2N} = O(1)$ a.s. under the condition $(C2)$. Hence

$$(4.23)\qquad \hat{\nu}_N - \nu_0 \stackrel{a.s.}{\to} 0$$

as $N \to \infty$. This implies the strong consistency of the estimator $\hat{\nu}_N$ under the conditions $(C1)$ and $(C2)$. Suppose that

(C3) $\frac{b_N}{\sum_{k=1}^N \beta_k} \stackrel{p}{\to} 1$ as $N \to \infty$.

Let $\delta_N = [\sum_{k=1}^N \beta_k]^{1/2}$. Applying Theorem 4.2, we get that

$$\begin{aligned}(4.24)\qquad \delta_N(\nu_N - \nu_0) &= [\delta_N]^{-1} \sum_{k=1}^N \alpha_k \\ &= \frac{\sum_{k=1}^N \lambda_k^{-\gamma+1} \int_0^T Q_k(t) dM_k^H(t)}{[\sum_{k=1}^N \lambda_k^{-2\gamma+2} \int_0^T E(Q_k^2(t)) dw_t^H]^{1/2}}\end{aligned}$$

and the last term converges in law to the standard normal distribution. Hence it follows that

$$(4.25)\qquad \delta_N(\hat{\nu}_N - \nu_0) \stackrel{\mathcal{L}}{\to} N(0,1) \ \text{as} \ N \to \infty$$

under the condition $(C3)$.

5 Estimation for 2D stochastic Navier-Stokes equation

Let P^N be the projection of the solution U of the stochastic Navier-Stokes equation (3.7) onto $J^N = P_N J \simeq R^N$. Then U^N satisfies the finite dimensional system

$$(5.1)\qquad dU^N = -(\nu A U^N + \psi^N)dt + P_N \sigma dW^H, U^N(0) = U_0^N$$

where $\psi^N(t) = P_N(B(U))$. To obtain an estimator of ν, we consider the equation as a stochastic differential equation in R^N driven by the multidimensional fractional Brownian motion $W^H \equiv (W_1^H, \ldots, W_N^H)$ with independent components. Following the arguments in Prakasa Rao (2003) (cf. Prakasa Rao (2010), p. 47), we can construct a corresponding stochastic differential system driven by the multidimensional Gaussian process $M^{N,H} = (M_1^H, \ldots, M_N^H)$ with independent martingale components each with the quadratic variation w^H. Following the definition (2.13), construct N-dimensional process $Q^{N,H}$ defined by

$$Q_i^{N,H}(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s)(\nu AU^N + \psi^N)_i [(P_N\sigma)^{-1}]_i dt \tag{5.2}$$

where $(P_N\sigma)^{-1} = diag(\sigma_1^{-1}, \ldots, \sigma_N^{-1}) = diag(\lambda_1^\gamma, \ldots, \lambda_N^\gamma)$. Here α' denotes the transpose of the vector α in R^N and $(\alpha)_i$ denotes the i-th component of the vector α in R^N. Note that

$$\begin{aligned} Q_i^{N,H}(t) &= \nu \frac{d}{dw_t^H} \int_0^t k_H(t,s)(AU^N)_i [(P_N\sigma)^{-1}]_i dt \\ &\quad + \frac{d}{dw_t^H} \int_0^t k_H(t,s)(\psi^N)_i [(P_N\sigma)^{-1}]_i dt. \\ &= \nu (J_{1N})_i(t) + (J_{2N})_i(t) \quad (say) \end{aligned} \tag{5.3}$$

Let $Q^{N,H} = (Q_1^{N,H}, \ldots, Q_N^{N,H})$. Let $Pr_\nu^{N,T}$ be the probability measure generated by the process U^N. It can be checked that the Radon-Nikodym derivative $\frac{dPr_\nu^{N,T}}{dPr_{\nu_0}^{N,T}}$ is given by

$$\begin{aligned} \log \frac{dPr_\nu^{N,T}}{dPr_{\nu_0}^{N,T}} &= (\nu - \nu_0) \int_0^T (J_{1N}(t))' dM^{N,H}(t) \\ &\quad - \frac{1}{2}(\nu - \nu_0)^2 \int_0^T [J_{1N}(t)]' J_{1N}(t) dw_t^H. \end{aligned} \tag{5.4}$$

and it is independent of the process J_{2N} and hence of the process ψ^N (cf. Prakasa Rao (2003, 2010)). Maximizing the likelihood ratio or the Radon-Nikodym derivative derived above with respect to the unknown parameter ν, we can obtain the maximum likelihood estimator ν_N of ν. Let ν_N be the maximum likelihood estimator. Furthermore, it can be checked that

$$\nu_N - \nu_0 = \frac{\int_0^T (J_{1N}(t))' dM^{N,H}(t)}{\int_0^T (J_{1N}(t))'(J_{1N}(t)) dw_t^H} = \frac{V_{1N}}{V_{2N}} \quad \text{(say)}. \tag{5.5}$$

Note that the term V_{1N} is a sum of independent random variables and $E[V_{2N}] = Var(V_{1N})$. Applying Theorems 4.1 and 4.2 again, it is possible to give sufficient conditions for the strong consistency and the asymptotic normality of the estimator ν_N.

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